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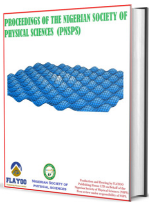

A novel class of Obrechkoff-type methods for solving boundary value problems

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ABSTRACT

A new family of high-order Obrechkoff-type block methods for the direct solution of second-order boundary value problems is proposed in this study. To discretize the differential equations, the methods are built using piecewise linear and constant approaches, as well as interpolation and collocation techniques based on Hermite splines of both first and higher orders. They are further developed using truncated power series expansions, achieving eighth-order accuracy with favourable stability properties. The method captures key characteristics of the problem and is computationally efficient. A thorough theoretical analysis confirms that it is consistent, convergent, and zero-stable. A single illustrative example is considered and comparison with the well-known Runge–Kutta method demonstrates its superiority in accuracy and error reduction.

Keywords: Obrechkoff-type methods, Boundary value problem, Second-order differential equation, Numerical method, Stability analysis.

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1. INTRODUCTION

Mathematics plays a fundamental role in the development of many disciplines, including biology, economics, chemical kinetics, circuit theory, and several other fields. In these areas, physical phenomena are often studied through mathematical modelling. Such models frequently lead to equations involving derivatives of unknown functions of one or more independent variables. These equations are known as differential equations.

Differential equations form the backbone of modelling problems in science, technology, engineering, and the social sciences. In most practical situations, exact solutions to these equations are either difficult or impossible to obtain analytically. Consequently, the development of numerical techniques for approximating their solutions becomes essential. A wide range of numerical methods has therefore been proposed, depending on the nature and class of the differential equations under consideration. These include finite difference methods, finite element methods, and finite volume methods, among others.

A boundary value problem (BVP) is made up of a differential equation and a set of extra constraints called boundary conditions. This study concentrates on the numerical solution of second-order boundary value problems of ordinary differential equations of the type

$$y'' = f(x, y, y'), \quad x \in [a, b], \tag{1}$$

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subject to boundary conditions imposed at the endpoints $x = a$ and $x = b$, such as

$$y(a) = y_b, \quad y(b) = y_N.$$

There are various categories into which boundary conditions can be divided. The values of the function itself at the borders are specified by a Dirichlet boundary condition (first-type condition), but the values of the function's derivative at the boundaries are specified by a Neumann boundary condition.

Numerical techniques for solving differential equations can broadly be categorised into linear multistep methods (LMMs) and one-step (single-step) methods [1–7]. Among these, Obrechkoff-type methods have proven to be particularly effective for both initial value problems (IVPs) and boundary value problems. The classical Numerov method represents one of the simplest forms of this class of methods. These approaches rely on incorporating higher-order derivatives in the formulation, which enhances both accuracy and stability.

For second-order problems, several high-order Obrechkoff-type methods have been developed in the literature. Consider the class of second-order boundary value problems with Dirichlet boundary conditions given by

$$y(a) = \alpha, \quad y(b) = \beta, \tag{2}$$

and Neumann boundary conditions given by

$$y'(a) = \alpha, \quad y'(b) = \beta. \tag{3}$$

Such problems arise in numerous applications across science, engineering, and medicine [8–14]. In addition, several authors have developed higher-order Obrechkoff-type schemes for second-order initial value problems [15, 16].

Obrechkoff [16] introduced, in 1942, a general class of methods for solving first-order initial value problems of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = \sum_{i=1}^{L} h^i \sum_{j=0}^{k} \beta_{ij} y^{(i)}_{n+j}. \tag{4}$$

2. DERIVATION OF THE BLOCK METHOD

To construct the proposed two-step Obrechkoff-type block scheme, the exact solution is locally approximated on each subinterval using a truncated power series expansion. Consider a partition of the interval $[a, b]$ given by:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n < x_{n+1} < \cdots < x_N = b.$$

On each subinterval, the approximate solution is assumed in the form of a power series polynomial

$$y(x) = \sum_{j=0}^{2s+r-1} a_j x^j, \tag{5}$$

where r and s denote the number of collocation and interpolation points, respectively.

Differentiating the above expression, the second derivative is obtained as:

$$y''(x) = \sum_{j=2}^{2s+r-1} j(j-1) a_j x^{j-2} = f(x, y), \tag{6}$$

and the fourth derivative is given by

$$y^{(4)}(x) = \sum_{j=4}^{2s+r-1} j(j-1)(j-2)(j-3) a_j x^{j-4} = \gamma(x, y). \tag{7}$$

Equations (5)-(7) serve as the interpolation and collocation conditions required for the construction of the method.

The resulting system of equations obtained from these conditions can be written in matrix form as:

$$AX = B, \tag{8}$$

where

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1s} \\ x_{21} & x_{22} & \cdots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{s1} & x_{s2} & \cdots & x_{ss} \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix}. \tag{9}$$

The algebraic system is solved using standard matrix techniques to determine the unknown coefficients a_j. Substituting these coefficients into the power series representation yields the continuous scheme. The continuous formulation is then evaluated at selected non-interpolating points to obtain the discrete block scheme. This process leads to the following system of equations:

$$\begin{aligned}
y_{n-1} &= -h^9 a_9 + h^8 a_8 - h^7 a_7 + h^6 a_6 - h^5 a_5 + h^4 a_4 \\
&\quad - h^3 a_3 + h^2 a_2 - h a_1 + a_0, \\
y_n &= a_0, \\
f_{n-1} &= -72h^7 a_9 + 56h^6 a_8 - 42h^5 a_7 + 30h^4 a_6 - 20h^3 a_5, \\
&\quad + 12h^2 a_4 - 6h a_3 + 2a_2 \\
f_n &= 2a_2, \\
f_{n+1} &= 72h^7 a_9 + 56h^6 a_8 + 42h^5 a_7 + 30h^4 a_6 + 20h^3 a_5 \\
&\quad + 12h^2 a_4 + 6h a_3 + 2a_2, \\
f_{n+2} &= 9216h^7 a_9 + 3584h^6 a_8 + 1344h^5 a_7 + 480h^4 a_6 \\
&\quad + 160h^3 a_5 + 48h^2 a_4 + 12h a_3 + 2a_2, \\
\gamma_{n-1} &= -3024h^5 a_9 + 1680h^4 a_8 - 840h^3 a_7 + 360h^2 a_6 \\
&\quad - 120h a_5 + 24a_4, \\
\gamma_n &= 24a_4, \\
\gamma_{n+1} &= 3024h^5 a_9 + 1680h^4 a_8 + 840h^3 a_7 + 360h^2 a_6 \\
&\quad + 120h a_5 + 24a_4, \\
\gamma_{n+2} &= 96768h^5 a_9 + 26880h^4 a_8 + 6720h^3 a_7 \\
&\quad + 1440h^2 a_6 + 240h a_5 + 24a_4.
\end{aligned} \tag{10}$$

Combining equations (8) and (10) in matrix form give:

$$X = \begin{bmatrix}
1 & -h & h^2 & -h^3 & h^4 & -h^5 & h^6 & -h^7 & h^8 & -h^9 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -6h & 12h^2 & -20h^3 & 30h^4 & -42h^5 & 56h^6 & -72h^7 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 6h & 12h^2 & 20h^3 & 30h^4 & 42h^5 & 56h^6 & 72h^7 \\
0 & 0 & 2 & 12h & 48h^2 & 160h^3 & 480h^4 & 1344h^5 & 3584h^6 & 9216h^7 \\
0 & 0 & 0 & 0 & 24 & -120h & 360h^2 & -840h^3 & 1680h^4 & -3024h^5 \\
0 & 0 & 0 & 0 & 24 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 24 & 120h & 360h^2 & 840h^3 & 1680h^4 & 3024h^5 \\
0 & 0 & 0 & 0 & 24 & 240h & 1440h^2 & 6720h^3 & 268880h^4 & 96768h^5
\end{bmatrix},$$

$$B = \begin{bmatrix} y_{n-1} \\ y_n \\ f_{n-1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ \gamma_{n-1} \\ \gamma_n \\ \gamma_{n+1} \\ \gamma_{n+2} \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}. \tag{11}$$

Applying the Gaussian elimination method to equation (11), the coefficients a_j, for $j = 0, 1, \ldots, 9$, are computed with the aid of Maple software as follows:

$$\begin{aligned}
a_0 &= y_n, \\
a_1 &= \frac{1}{90720h}\Big(13701h^4\gamma_n + 434h^4\gamma_{n-1} + 9480h^4\gamma_{n+1} \\
&\quad + 505h^4\gamma_{n+2} + 61164h^2 f_n - 4488h^2 f_{n-1} - 3024h^2 f_{n+1} \\
&\quad - 8292h^2 f_{n+2} + 90720y_n - 90720y_{n-1}\Big), \\
a_2 &= \frac{1}{2} f_n, \\
a_3 &= -\frac{1}{3780h}\Big(772h^2\gamma_n + 36h^2\gamma_{n-1} + 628h^2\gamma_{n+1} + 34h^2\gamma_{n+2} \\
&\quad + 1275f_n - 600f_{n-1} - 120f_{n+1} - 555f_{n+2}\Big), \\
a_4 &= \frac{1}{4}\gamma_n, \\
a_5 &= \frac{1}{720h^4}\Big(61h^2\gamma_n + 4h^2\gamma_{n-1} + 52h^2\gamma_{n+1} + 3h^2\gamma_{n+2} \\
&\quad + 96f_n - 72f_{n-1} + 24f_{n+1} - 48f_{n+2}\Big), \\
a_6 &= \frac{1}{1080h^4}\Big(28h^2\gamma_n + h^2\gamma_{n-1} + h^2\gamma_{n+1} + 60f_n - 30f_{n-1}, \\
&\quad - 30f_{n+1}\Big), \\
a_7 &= -\frac{1}{2520h^5}\Big(26h^2\gamma_n + 3h^2\gamma_{n-1} + 29h^2\gamma_{n+1} + 2h^2\gamma_{n+2} \\
&\quad + 30f_n - 30f_{n-1} + 30f_{n+1} - 30f_{n+2}\Big), \\
a_8 &= \frac{1}{2016h^6}\Big(10h^2\gamma_n + h^2\gamma_{n-1} + h^2\gamma_{n+1} + 24f_n - 12f_{n-1} \\
&\quad - 12f_{n+1}\Big), \\
a_9 &= -\frac{1}{18144h^7}\Big(9h^2\gamma_n + h^2\gamma_{n-1} - 9h^2\gamma_{n+1} - h^2\gamma_{n+2} \\
&\quad + 36f_n - 12f_{n-1} - 36f_{n+1} + 12f_{n+2}\Big).
\end{aligned} \tag{12}$$

Upon substituting the coefficients a_j into equation (5), a linear multistep method with continuous coefficients is obtained in the form:

$$\begin{aligned}
y(x) &= \alpha_{-1} y_{n-1} + \alpha_0 y_n \\
&\quad + h^2\left(\beta_{-1} f_{n-1} + \beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}\right) \\
&\quad + h^4\left(g_{-1}\gamma_{n-1} + g_0\gamma_n + g_1\gamma_{n+1} + g_2\gamma_{n+2}\right).
\end{aligned} \tag{13}$$

The coefficients associated with equation (13) are derived using the following transformation:

$$x_{n+k-1} = x - ht, k = 2, \tag{14}$$

where

$$\begin{aligned}
\alpha_{-1}(t) &= -t, \\
\alpha_0(t) &= 1 + t, \\
\beta_{-1}(t) &= -\frac{1}{3780}h^2 t\left(45t^6 + 105t^5 - 378t^4 + 600t^2 - 187\right), \\
\beta_0(t) &= \frac{1}{210}h^2 t\left(3t^6 - 7t^5 - 21t^2 + 105t + 122\right), \\
\beta_1(t) &= -\frac{1}{2520}h^2 t\Big(30t^6 + 140t^5 - 336t^4 + 850t^2 \\
&\quad - 1260t - 1699\Big), \\
\beta_2(t) &= \frac{1}{7560}h^2 t\left(90t^6 - 504t^4 + 1110t^2 - 691\right), \\
g_{-1}(t) &= -\frac{1}{45360}h^4 t\left(54t^6 + 42t^5 - 252t^4 + 432t^2 - 217\right), \\
g_0(t) &= -\frac{1}{30240}h^4 t\Big(312t^6 + 784t^5 - 2562t^4 - 1260t^3 \\
&\quad + 6176t^2 - 4567\Big), \\
g_1(t) &= -\frac{1}{7560}h^4 t(t+1)\Big(87t^5 - 80t^4 - 466t^3 + 466t^2 \\
&\quad + 790t - 790\Big), \\
g_2(t) &= -\frac{1}{90720}h^4 t\left(72t^6 - 378t^4 + 816t^2 - 505\right).
\end{aligned} \tag{15}$$

Evaluating equation (13) at $t = 2$, gives the discrete scheme below:

$$\begin{aligned}
y_{n+2} &= 3y_n - 2y_{n-1} + \frac{59}{720}h^4\gamma_n - \frac{13}{7560}h^4\gamma_{n-1} + \frac{5}{126}h^4\gamma_{n+1} \\
&\quad - \frac{13}{15120}h^4\gamma_{n+2} + \frac{157}{84}h^2 f_n + \frac{11}{126}h^2 f_{n-1} \\
&\quad + h^2 f_{n+1} + \frac{11}{252}h^2 f_{n+2}
\end{aligned} \tag{16}$$

and equation (13) at $t = 1$ to get the additional method:

$$\begin{aligned}
y_{n+1} &= 2y_n - y_{n-1} + \frac{313}{7560}h^4\gamma_n - \frac{13}{15120}h^4\gamma_{n-1} - \frac{13}{15120}h^4\gamma_{n+1} \\
&\quad + \frac{115}{126}h^2 f_n + \frac{11}{252}h^2 f_{n-1} + \frac{11}{252}h^2 f_{n+1}.
\end{aligned} \tag{17}$$

The first derivatives of equation (13) gives:

$$\begin{aligned}
y'(x) &= \alpha'_{-1} y_{n-1} + \alpha'_0 y_n \\
&\quad + h^2\left(\beta'_{-1} f_{n-1} + \beta'_0 f_n + \beta'_1 f_{n+1} + \beta'_2 f_{n+2}\right) \\
&\quad + h^4\left(g'_{-1}\gamma_{n-1} + g'_0\gamma_n + g'_1\gamma_{n+1} + g'_2\gamma_{n+2}\right),
\end{aligned} \tag{18}$$

where

$$
\begin{aligned}
\alpha'_{-1}(t) &= -1,\\
\alpha'_{0}(t) &= 1,\\
\beta'_{-1}(t) &= -\frac{h^2}{3780}\left(315t^6 + 630t^5 - 1890t^4 + 1800t^2 - 187\right),\\
\beta'_{0}(t) &= \frac{h^2}{2520}\Big(210t^6 + 840t^5 - 1680t^4 - 2550t^2 - 2520t,\\
&\quad - 1699\Big),\\
\beta'_{1}(t) &= \frac{h^2}{420}\left(35t^6 - 140t^5 - 70t^2 + 14\right),\\
\beta'_{2}(t) &= \frac{h^2}{7560}\left(90t^6 - 504t^4 + 1110t^2 - 691\right),\\
g'_{-1}(t) &= \frac{h^4}{45360}\left(378t^6 + 252t^5 - 1260t^4 + 1296t^2 - 217\right),\\
g'_{0}(t) &= \frac{h^4}{30240}\Big(2184t^6 + 470t^5 - 12610t^4 - 5040t^3\\
&\quad + 18528t^2 - 4567\Big),\\
g'_{1}(t) &= \frac{h^4}{7560}\left(609t^6 + 42t^5 - 2730t^4 + 768t^2 - 790\right),\\
g'_{2}(t) &= \frac{h^4}{90720}\left(504t^6 - 1890t^4 + 2448t^2 - 505\right).
\end{aligned}
\tag{19}
$$

Evaluating equation (18) at $t = -1, 0, 1,$ and 2 yields the following results:

$$
\begin{aligned}
y'_{n-1} = -\frac{1}{90720h}&(15018h^4\gamma_n + 295h^4\gamma_{n-1} + 9735h^4\gamma_{n+1}\\
&+ 512h^4\gamma_{n+2} + 48456h^2f_n + 9348h^2f_{n-1}\\
&- 3996h^2f_{n+1} - 8448h^2f_{n+2} - 90720y_n\\
&+ 90720y_{n-1}),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
y'_{n} = \frac{1}{90720h}&(13701h^4\gamma_n + 434h^4\gamma_{n-1} + 9480h^4\gamma_{n+1}\\
&+ 505h^4\gamma_{n+2} + 61164h^2f_n - 4488h^2f_{n-1}\\
&- 3024h^2f_{n+1} - 8292h^2f_{n+2} + 90720y_n - 90720y_{n-1}),
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
y'_{n+1} = -\frac{1}{90720h}&(5802h^4\gamma_n + 583h^4\gamma_{n-1} + 10023h^4\gamma_{n+1}\\
&+ 512h^4\gamma_{n+2} - 89784h^2f_n - 12252h^2f_{n-1} - 25596h^2f_{n+1}\\
&- 8448h^2f_{n+2} - 90720y_n + 90720y_{n-1}),
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
y'_{n+2} = \frac{1}{90720h}&(13413h^4\gamma_n + 434h^4\gamma_{n-1} + 18696h^4\gamma_{n+1}\\
&+ 217h^4\gamma_{n+2} + 82764h^2f_n - 4488h^2f_{n-1} + 135216h^2f_{n+1}\\
&+ 13308h^2f_{n+2} + 90720y_n - 90720y_{n-1}).
\end{aligned}
\tag{23}
$$

3. BLOCK FORM OF TWO-STEP OBRECHKOFF-TYPE METHOD

By combining equations (16), (17) and (18), the resulting block method is obtained as follows:

$$
\begin{aligned}
&\begin{bmatrix} 2 & 0 & 1\\ 1 & 1 & 0\\ 1 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n-1}\\ y_{n+1}\\ y_{n+2} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 3\\ 0 & 0 & 2\\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_n \end{bmatrix}
+ h\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1 \end{bmatrix}
\begin{bmatrix} y' \end{bmatrix}.\\
&+h^2\begin{bmatrix} 0 & 0 & \frac{157}{84}\\ 0 & 0 & \frac{115}{126}\\ 0 & 0 & \frac{61164}{90720} \end{bmatrix}
\begin{bmatrix} f_n \end{bmatrix}
+ h^2\begin{bmatrix} \frac{11}{126} & \frac{11}{126} & \frac{11}{252}\\ \frac{11}{252} & \frac{11}{252} & 0\\ -\frac{4488}{90720} & -\frac{3024}{90720} & -\frac{8292}{90720} \end{bmatrix}
\begin{bmatrix} f_{n-1}\\ f_{n+1}\\ f_{n+2} \end{bmatrix}\\
&+h^4\begin{bmatrix} 0 & 0 & \frac{59}{720}\\ 0 & 0 & \frac{313}{7560}\\ 0 & 0 & \frac{13701}{90720} \end{bmatrix}
\begin{bmatrix} \gamma_n \end{bmatrix}
+ h^4\begin{bmatrix} -\frac{13}{7560} & \frac{5}{126} & -\frac{13}{15120}\\ -\frac{13}{15120} & -\frac{13}{15120} & 0\\ \frac{434}{90720} & \frac{9480}{90720} & \frac{505}{90720} \end{bmatrix}
\begin{bmatrix} \gamma_{n-1}\\ \gamma_{n+1}\\ \gamma_{n+2} \end{bmatrix}.
\end{aligned}
\tag{24}
$$

Solving equation (24) using matrix inversion gives:

$$
\begin{aligned}
&\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_{n-1}\\ y_{n+1}\\ y_{n+2} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} y_n \end{bmatrix}
+ h\begin{bmatrix} 0 & 0 & -1\\ 0 & 0 & 1\\ 0 & 0 & 2 \end{bmatrix}
\begin{bmatrix} y'_n \end{bmatrix}\\
&+h^2\begin{bmatrix} 0 & 0 & \frac{1699}{2520}\\ 0 & 0 & \frac{601}{2520}\\ 0 & 0 & \frac{164}{315} \end{bmatrix}
\begin{bmatrix} f_n \end{bmatrix}
+ h^2\begin{bmatrix} -\frac{187}{3780} & -\frac{1}{30} & -\frac{691}{7560}\\ \frac{88}{945} & \frac{97}{1260} & \frac{691}{7560}\\ \frac{176}{945} & \frac{16}{15} & \frac{214}{945} \end{bmatrix}
\begin{bmatrix} f_{n-1}\\ f_{n+1}\\ f_{n+2} \end{bmatrix}\\
&+h^4\begin{bmatrix} 0 & 0 & \frac{4567}{30240}\\ 0 & 0 & -\frac{221}{2016}\\ 0 & 0 & -\frac{208}{945} \end{bmatrix}
\begin{bmatrix} \gamma_n \end{bmatrix}
+ h^4\begin{bmatrix} \frac{31}{6480} & \frac{79}{756} & \frac{101}{18144}\\ -\frac{16}{2835} & -\frac{59}{560} & -\frac{101}{18144}\\ -\frac{32}{2835} & -\frac{32}{189} & -\frac{34}{2835} \end{bmatrix}
\begin{bmatrix} \gamma_{n-1}\\ \gamma_{n+1}\\ \gamma_{n+2} \end{bmatrix}.
\end{aligned}
\tag{25}
$$

Writing out equation (25) explicitly gives:

$$
\begin{aligned}
y_{n-1} ={}& \frac{4567}{30240}h^4\gamma_n + \frac{31}{6480}h^4\gamma_{n-1} + \frac{79}{756}h^4\gamma_{n+1} + \frac{101}{18144}h^4\gamma_{n+2} \\
&+ \frac{1699}{2520}h^2f_n - \frac{187}{3780}h^2f_{n-1} - \frac{1}{3}h^2f_{n+1} - \frac{691}{7560}h^2f_{n+2} \\
&+ y_n - hy'_n, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+1} ={}& -\frac{221}{2016}h^4\gamma_n - \frac{16}{2835}h^4\gamma_{n-1} - \frac{59}{560}h^4\gamma_{n+1} - \frac{101}{18144}h^4\gamma_{n+2} \\
&+ \frac{601}{2520}h^2f_n + \frac{88}{945}h^2f_{n-1} + \frac{97}{1260}h^2f_{n+1} + \frac{691}{7560}h^2f_{n+2} \\
&+ y_n + hy'_n, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
y_{n+2} ={}& -\frac{208}{945}h^4\gamma_n - \frac{32}{2835}h^4\gamma_{n-1} - \frac{32}{189}h^4\gamma_{n+1} - \frac{34}{2835}h^4\gamma_{n+2} \\
&+ \frac{164}{315}h^2f_n + \frac{176}{945}h^2f_{n-1} + \frac{16}{15}h^2f_{n+1} + \frac{214}{945}h^2f_{n+2} \\
&+ y_n + 2hy'_n. \qquad (28)
\end{aligned}
$$

Substituting equations (26), (27) and (28) into equations (20), (22) and (23) yields:

$$
\begin{aligned}
y'_{n-1} ={}& -\frac{3191}{10080}h^3\gamma_n - \frac{9}{1120}h^3\gamma_{n-1} - \frac{61}{288}h^3\gamma_{n+1} \\
&- \frac{113}{10080}h^3\gamma_{n+2} - \frac{29}{24}hf_n - \frac{3}{56}hf_{n-1} + \frac{13}{168}hf_{n+1} \\
&+ \frac{31}{168}hf_{n+2} + y'_n, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+1} ={}& -\frac{2167}{10080}h^3\gamma_n - \frac{113}{10080}h^3\gamma_{n-1} - \frac{2167}{10080}h^3\gamma_{n+1} \\
&- \frac{113}{10080}h^3\gamma_{n+2} + \frac{53}{168}hf_n + \frac{31}{168}hf_{n-1} + \frac{53}{168}hf_{n+1} \\
&+ \frac{31}{168}hf_{n+2} + y'_n, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
y'_{n+2} ={}& -\frac{1}{315}h^3\gamma_n - \frac{1}{315}h^3\gamma_{n-1} + \frac{32}{315}h^3\gamma_{n+1} - \frac{1}{315}h^3\gamma_{n+2} \\
&+ \frac{5}{21}hf_n + \frac{32}{21}hf_{n+1} + \frac{5}{21}hf_{n+2} + y'_n. \qquad (31)
\end{aligned}
$$

4. ANALYSIS OF THE METHOD

In this section, we examine the derived scheme's order and error constants, consistency, zero stability, convergence, and absolute stability region.

4.1. LOCAL TRUNCATION ERROR AND ORDER

The order of the derived system was determined using the methodology described by Ref. [17]. Equation (31) gives the error constants, showing the method has order $P = 8$:

$$C_{10} = \left[\tfrac{59}{25401600}\right].$$

The block method also has order $P = 8$, with the corresponding error constants represented as:

$$C_{p+2} = \left[-\frac{9743}{152409600}, \frac{3287}{50803200}, \frac{31}{238140}\right]^T.$$

4.2. CONSISTENCY

A linear multistep method is consistent if it converges with an order of at least 2, i.e., $p \geq 2$.

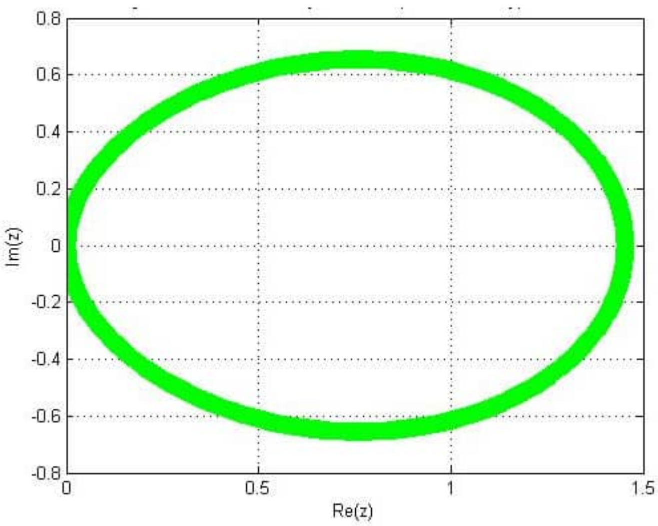

Figure 1. Region of absolute stability of the method.

Table 1. Exact solutions, computed results, and errors from the proposed 2SOBM method.

x	y_{exact}	y_{computed}	Error in (2SOBM)
0.1	-0.205645689	-0.2056456887	3.0×10^{-10}
0.2	-0.357615417	-0.3576154167	3.0×10^{-10}
0.3	-0.462008263	-0.4620082634	4.0×10^{-10}
0.4	-0.523013879	-0.5230138797	7.0×10^{-10}
0.5	-0.543080635	-0.5430806358	8.0×10^{-10}
0.6	-0.523013879	-0.5230138803	1.3×10^{-9}
0.7	-0.462008263	-0.4620082646	1.6×10^{-9}
0.8	-0.357615417	-0.3576154186	1.6×10^{-9}
0.9	-0.205645689	-0.2056456912	2.2×10^{-9}
1.0	0.000000000	3.3×10^{-9}	3.3×10^{-9}

4.3. CONVERGENCE

Consistency together with zero stability constitutes the necessary and sufficient conditions for the method to be convergent [17].

4.4. ZERO STABILITY

A linear multistep method is *zero-stable* for any well-behaved initial value problem if:

- all roots of the first characteristic polynomial $\rho(r) = 0$ lie within or on the unit circle, i.e., $|r| \leq 1$;
- any root on the unit circle ($|r| = 1$) is simple (i.e., has multiplicity one) [17, 18].

The characteristic polynomial of the block method is:

$$P(z) = \det\left[z\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\end{bmatrix}\right]. \qquad (32)$$

Evaluating the determinant gives:

$$P(z) = \det\begin{bmatrix}z & 0 & -1\\ 0 & z & -1\\ 0 & 0 & z-1\end{bmatrix} = z^2(z-1). \qquad (33)$$

Solving $P(z) = 0$ yields the roots $z = 0$ (double root) and $z = 1$. Since all roots satisfy $|z| \leq 1$, and the only root on the unit

Table 2. Comparison of exact solution, 2SOBM, and RK4 results with errors.

x	y_{exact}	y_{2SOBM}	Error (2SOBM)	y_{RK4}	Error (RK4)
0.1	-0.205645689	-0.2056456887	3.00×10^{-10}	-0.20564558	1.09×10^{-7}
0.2	-0.357615417	-0.3576154167	3.00×10^{-10}	-0.35761531	1.07×10^{-7}
0.3	-0.462008263	-0.4620082634	4.00×10^{-10}	-0.46200816	1.03×10^{-7}
0.4	-0.523013879	-0.5230138797	7.00×10^{-10}	-0.52301376	1.19×10^{-7}
0.5	-0.543080635	-0.5430806358	8.00×10^{-10}	-0.54308052	1.15×10^{-7}
0.6	-0.523013879	-0.5230138803	1.30×10^{-9}	-0.52301377	1.09×10^{-7}
0.7	-0.462008263	-0.4620082646	1.60×10^{-9}	-0.46200815	1.13×10^{-7}
0.8	-0.357615417	-0.3576154186	1.60×10^{-9}	-0.35761530	1.17×10^{-7}
0.9	-0.205645689	-0.2056456912	2.20×10^{-9}	-0.20564557	1.20×10^{-7}
1.0	0.000000000	3.30×10^{-9}	3.30×10^{-9}	2.50×10^{-8}	2.50×10^{-8}

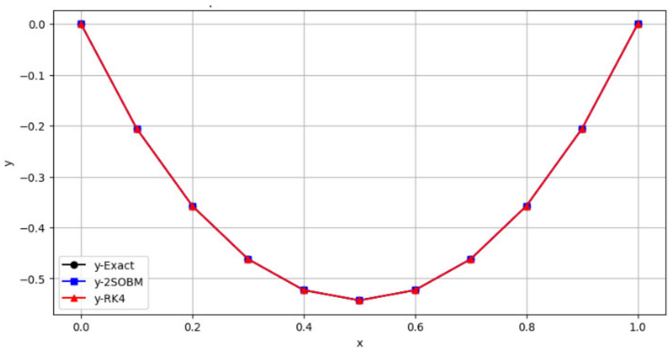

Figure 2. The comparative study of exact solution, 2SOBM and RK4 using Table 2.

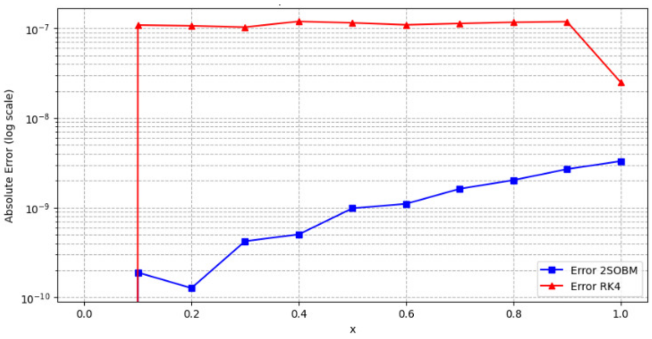

Figure 3. Absolute errors incurred in 2SOBM and RK4 using Table 2.

Table 3. Maximum and final absolute errors for $h = 0.1$

Method	Maximum absolute error	Final error at $x = 1$
2SOBM	2.2×10^{-9}	3.3×10^{-9}
RK4	1.2×10^{-7}	2.5×10^{-8}

circle ($z = 1$) is simple, the method satisfies the zero-stability condition. Hence, the block method is zero-stable.

4.5. REGION OF ABSOLUTE STABILITY

Following [18], the region of absolute stability (RAS) of the scheme is determined to assess the method's stability behaviour (Figure 1).

Table 4. Maximum absolute errors and empirical orders of convergence for 2SOBM and RK4 at different step sizes h.

h	Max Error (2SOBM)	Order (2SOBM)	Max Error (RK4)	Order (RK4)
1.0×10^{-1}	2.20×10^{-9}	–	1.20×10^{-7}	–
5.0×10^{-2}	8.59×10^{-12}	8	7.50×10^{-9}	4
2.5×10^{-3}	3.36×10^{-14}	8	4.69×10^{-10}	4
1.25×10^{-2}	1.31×10^{-16}	8	2.93×10^{-11}	4
6.25×10^{-3}	5.12×10^{-19}	8	1.83×10^{-12}	4

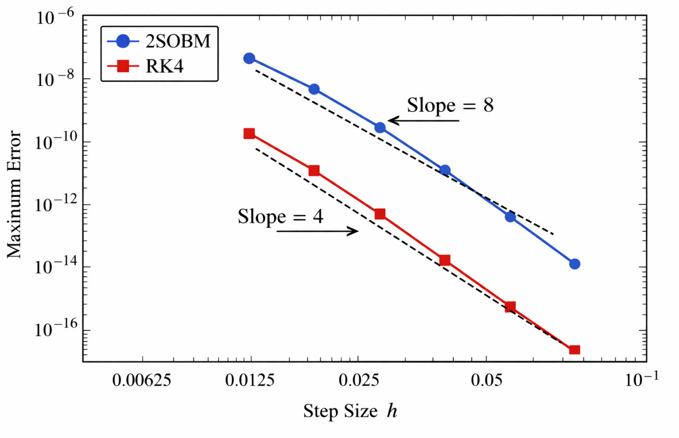

Figure 4. The plot of Table 4.

5. NUMERICAL APPLICATION

In order to evaluate the effectiveness of the produced methods, the new method's actual performance is assessed on a few test cases in this part.

5.1. ILLUSTRATIVE EXAMPLE

Consider the boundary value problem [19]:

$$y'' - 4y = 4\cosh(1), \quad y(0) = 0, \quad y(1) = 0, \quad h = \frac{1}{10}, \tag{34}$$

with the exact solution given by:

$$y(x) = \cosh(2x - 1) - \cosh(1). \tag{35}$$

The comparative results are presented in Table 1.

5.2. DISCUSSION OF RESULTS

Tables 1 and 2 (also Figures 2 and 3) indicate that the numerical values obtained with the Two-Step Optimal Boundary Method (2SOBM) are very close to the exact solution across the entire interval. The corresponding absolute errors remain very small, generally within the range of 10^{-10} to 10^{-9}. Although there is a slight increase in error as x approaches 1, the change is very minimal and hardly noticeable. This small variation does not affect the overall accuracy; instead, it reflects the stability of the method. The difference in performance between the two methods remains clear throughout the interval, showing that the proposed scheme consistently maintains a high level of accuracy. This observation agrees with the results presented in Table 3. For instance, when $h = 0.1$, the maximum error for 2SOBM is 2.2×10^{-9}, whereas RK4 produces a larger error of 1.2×10^{-7}. A similar pattern is observed at the final grid point, confirming that the improved accuracy is maintained throughout the domain. Table 4 (and Figure 4) further supports this by comparing the convergence behavior of both methods. The 2SOBM demonstrates eighth-order convergence, with errors decreasing rapidly as the step size becomes smaller, reaching values as low as 10^{-19} when $h = 0.00625$. In contrast, RK4 follows its classical fourth-order convergence, with a slower reduction in error and values around 10^{-12} at the same step size. Taken together, these results show that 2SOBM can achieve high accuracy even with relatively coarse step sizes, requiring fewer refinements to reach very small error levels. This makes it an efficient and dependable approach, especially for problems where a high level of precision is required.

6. CONCLUSION

A novel class of two-step Obrechkoff-type block methods for the direct solution of second-order boundary value problems is presented in this study. Both Dirichlet and Neumann boundary conditions can be used with these techniques. The interpolation and collocation of power series expansions serve as the foundation for their derivation. The schemes are consistent, zero-stable, and convergent with an overall order of eight, according to theoretical study. Additionally, numerical experiments show greater accuracy with less absolute errors and faster convergence rates when compared to traditional methods like the Runge-Kutta method. These outcomes demonstrate how well the suggested techniques work for resolving boundary value issues. Future research may expand the methodology to include nonlinear systems, higher-dimensional challenges, and real-world physics and engineering applications.

DATA AVAILABILITY

The data will be available on request from the corresponding author.

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