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Estimates for a class of analytic and univalent functions connected with quasi-subordination

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ABSTRACT

Geometric Function Theory, an active field of study that has its roots in complex analysis, has gained an impressive attention from many researchers. This occurs largely because it deals with the study of geometric properties of analytic (and univalent) functions where many of its applications spread across many fields of mathematics, mathematical physics and engineering. Notable areas of application include conformal mappings, special functions, orthogonal polynomials, fluid flows in physics, and engineering designs. The investigations in this paper are on a subclass of analytic and univalent functions defined in the unit disk Ω and denoted by $\mathbb{Q}_q^\alpha(m)$. The definition of the new class encompasses some well-known subclasses of analytic and univalent functions such as the classes of starlike functions, Yamaguchi functions, and Ma-Minda functions. Two key mathematical principles involved in the definition of the class are the principles of Taylor’s series and quasi-subordination. Some of the investigations carried out on functions $f \in \mathbb{Q}_q^\alpha(m)$ are however, the upper estimates for some initial bounds, the solution to the well-known Fekete-Szegő problem and the upper estimate for a Hankel determinant.

Keywords: Starlike function, Ma-Minda function, Yamaguchi function, Fekete-Szegő functional.

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1. INTRODUCTION

Let \mathbb{A} denote the class of complex-valued functions f which are analytic in the open unit disk

$$\{z \mid z \in \mathbb{C} \text{ and } |z| < 1\} = \Omega.$$

Let \mathbb{S} be a subclass of \mathbb{A} and consists of functions of the Taylor’s series form

$$f(z) = z + a_2z^2 + a_3z^3 + \dots + a_rz^r + \dots$$

$$(f(0) = f'(0) - 1 = 0, z \in \Omega). \tag{1}$$

Class \mathbb{S} is known as the class of normalized analytic and univalent functions defined in Ω . Some known subclasses of \mathbb{S} are the classes of starlike functions (whenever $\Re(zf'/f) > 0$), convex functions (whenever $\Re[1 + (zf''/f)'] > 0$), close-to-convex functions (whenever $\Re(zf'/g) > 0$, g is starlike), close-to-star functions (whenever $\Re(f/g) > 0$, g is starlike), bounded turning functions (whenever $\Re(f') > 0$), Yamaguchi functions (whenever $\Re(f/z) > 0$) and many more. See Refs. [1–3] for more details.

The foundation of coefficient problems in the theory of normalized analytic and univalent functions of the series form in Eq. (1) can be ascribed to Bieberbach’s conjecture [1, 4] of 1916 where it was conjectured that $|a_r| \leq r, \forall r \in \{2, 3, 4, \dots\}$. In Ref.

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[4], Duren affirmed that *coefficient problem* is the determination of that part of the $(n - 1)$ -dimensional complex plane that are occupied by the points $(a_2, a_3, a_4, \dots, a_r)$ of function f . In 1985, Branges [5] validated the conjecture to be true and this affirmation elevated the field of Geometric Function Theory to one of the ever growing areas of possible research. Some coefficient problems that have been persistently studied include the upper estimates for coefficient bounds, Fekete-Szegő functional, Hankel determinants and Toeplitz determinants. Some recent studies in this regard are accessible in Refs. [6–12].

For any two analytic functions f and h , the function f is *subordinate* to h , expressed as

$$f(z) < h(z) \quad (z \in \Omega)$$

if there exists an analytic function

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \dots + \omega_r z^r + \dots$$

$$(|\omega(z)| < 1, \omega(0) = 0, z \in \Omega), \quad (2)$$

such that

$$f(z) = h(\omega(z)).$$

Suppose h is a univalent function defined in Ω , then

$$f(z) < h(z) \quad \text{if and only if} \quad f(0) = h(0) \quad \text{and} \quad f(\Omega) \subset h(\Omega).$$

Robertson [13] introduced the notion of quasi-subordination as follows. For two analytic functions f and h , the function f is *quasi-subordinate* to h , expressed as

$$f(z) <_q h(z) \quad (z \in \Omega) \quad (3)$$

if there exist the analytic functions ω in Eq. (2) and

$$\varpi(z) = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \dots + A_r z^r + \dots$$

$$(|\varpi(z)| \leq 1, z \in \Omega), \quad (4)$$

such that

$$f(z) = \varpi(z)h(\omega(z)) \quad (z \in \Omega).$$

According to MacGregor [14], suppose $\varpi(z) = 1$, then $f(z) = h(\omega(z))$, so that $f(z) < h(z) \quad (z \in \Omega)$. Also, if $\omega(z) = z$, then $f(z) = \varpi(z)h(z)$ and it is said that f is *majorized* by h , expressed as $f(z) \ll h(z)$. Note that quasi-subordination is a generalization of both subordination and majorization concepts. For some related works on quasi-subordination, see Refs. [15–17].

Using the notion of subordination, Ma and Minda [18] defined and studied the classes

$$\mathbb{S}^*(m) = \left\{ f : f \in \mathbb{A}, \frac{zf'(z)}{f(z)} < m(z), \text{ and } z \in \Omega \right\}$$

and

$$\mathbb{K}(m) = \left\{ f : f \in \mathbb{A}, 1 + \frac{zf''(z)}{f'(z)} < m(z), \text{ and } z \in \Omega \right\},$$

where

$$m(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots + B_r z^r + \dots$$

$$(B_1 \in \mathbb{R}^+, z \in \Omega) \quad (5)$$

is an analytic function with positive real part in Ω , starlike with respect to 1, and symmetric with respect to the real axis; so that $m(0) = 1$ and $m'(0) > 0$. The function $m(z)$ has been found to amalgamate several functions whose real parts are positive. Some recent instances are present in Refs. [6, 7, 9, 11, 16, 17, 19] and many more.

In 1966, Pommerenke [12] demonstrated that the t -th Hankel determinant for the coefficients of function f in Eq. (1) is

$$\Delta_{t,r}(f) = \begin{vmatrix} 1 & a_{r+1} & \dots & a_{r+t-1} \\ a_{r+1} & a_{r+2} & \dots & a_{r+t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+t-1} & a_{r+t} & \dots & a_{r+2(t-1)} \end{vmatrix}, \quad (6)$$

where $a_1 = 1, t \geq 1$, and $r \geq 1$. This determinant has been considered for specific values of r and t by many authors. For instance, see Refs. [8, 9, 11, 19]. It has also been established that the Fekete-Szegő functional

$$\nabla(\gamma, f) = |a_3 - \gamma a_2^2| \quad (7)$$

and the Hankel determinant in Eq. (6) are related such that

$$|\Delta_{2,1}(f)| = |a_3 - a_2^2| = \nabla(1, f).$$

For some recent works on Fekete-Szegő functional in Eq. (7), see Refs. [2, 6–8, 10, 11].

2. LEMMAS

The following lemmas shall be needed to prove our results. Firstly, let \mathbb{P} denote the class of Carathéodory functions of the series type

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots + c_r z^r + \dots \quad (z \in \Omega) \quad (8)$$

which are analytic and satisfy the conditions: $p(0) = 1$ and $\Re p(z) > 0$.

Lemma 2.1 ([1, 3]). *Let $p \in \mathbb{P}$. Then $|c_r| \leq 2, \forall r \in \mathbb{N}$.*

Lemma 2.2 ([20]). *Let $p \in \mathbb{P}$. Then for any real number λ ,*

$$\left| c_2 - \lambda \frac{c_1^2}{2} \right| \leq \begin{cases} 2(1 - \lambda) & \text{if } \lambda \leq 0 \\ 2 & \text{if } 0 \leq \lambda \leq 2 \\ 2(\lambda - 1) & \text{if } \lambda \geq 2 \end{cases}$$

and for any complex number λ ,

$$\left| c_2 - \lambda \frac{c_1^2}{2} \right| \leq 2 \max \{1, |1 - \lambda|\}.$$

3. MAIN RESULTS

Definition 3.1. A function $f \in \mathbb{A}$ is said to be a member of class $\mathbb{Q}_q^\alpha(m)$ if the geometric condition

$$\left(\frac{zf'(z)}{f(z)} \right) \left(\frac{f(z)}{z} \right)^\alpha - 1 <_q (m(z) - 1) \quad (9)$$

holds for $\alpha \geq 0, m$ in Eq. (5) and $z \in \Omega$.

Remark 3.2. The following are some of the subclasses of the new class $\mathbb{Q}_q^\alpha(m)$.

1. If $\alpha = 0$, then the geometric expressions on the LHS of Eq. (9) will reduce to the geometric expression for starlike functions discussed in Refs. [1, 3].
2. If $\alpha = 1$, then the geometric expressions on the LHS of Eq. (9) will reduce to the *product combination* of geometric expressions for starlike functions in Refs. [1, 3] and Yamaguchi functions in Ref. [21].

Example 3.3. For example, if $\alpha = 2$, then condition (9) will reduce to the analytic functions

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^2 - 1 <_q (m(z) - 1)$$

$$\frac{f'(z)f(z)}{z} - 1 <_q (m(z) - 1) \tag{10}$$

and if we substitute Eqs. (1) and (5) into Eq. (10), then we get:

$$3a_2z + 2(a_2^2 + 2a_3)z^2 + 5(a_2a_3 + a_4)z^3 +$$

$$3(2a_2a_4 + a_3^2 + 3a_5)z^4 + \dots <_q B_1z + B_2z^2 + B_3z^3 + B_4z^4 + \dots$$

Notably, functions $f \in \mathbb{Q}_q^\alpha(m)$ are associated with the class of functions with positive real parts. The objectives of this paper is to investigate the upper estimates for the bounds on $|a_r|$ ($r \in \{2, 3\}$), the Fekete-Szegő functional $|a_3 - \gamma a_2^2|$ and the Hankel determinant $|\Delta_{2,1}(f)|$ for functions $f \in \mathbb{Q}_q^\alpha(m)$.

Theorem 3.4. *If $f \in \mathbb{Q}_q^\alpha(m)$ is given by Eq. (1), then*

$$|a_2| \leq \frac{|A_0B_1|}{1 + \alpha} \tag{11}$$

and

$$|a_3| \leq \frac{|A_1|B_1 + |A_0|B_1 + |A_0||B_2| + \frac{|\alpha-1|(\alpha+2)|A_0^2|B_1^2}{4(1+\alpha)^2}}{2 + \alpha} \tag{12}$$

Proof. Let $f \in \mathbb{Q}_q^\alpha(m)$, then there exist the analytic functions ω , ϖ and m in Eqs. (2), (4), and (5), respectively; such that Eq. (9) can be expressed as

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\alpha - 1 = \varpi(z)(m(\omega(z)) - 1). \tag{13}$$

Considering the RHS of Eq. (13) means that

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left\{ c_1z + \left[\left(c_2 - \frac{c_1^2}{2} \right) + \frac{c_1^2}{2} \right] z^2 \right.$$

$$\left. + \left[\left(c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + c_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{c_1^3}{8} \right] z^3 + \dots \right\}$$

and

$$\varpi(z) \left(m \left(\frac{p(z) - 1}{p(z) + 1} \right) - 1 \right) = \frac{A_0B_1c_1}{2} z +$$

$$\left[\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{A_0B_2}{4} c_1^2 \right] z^2$$

$$+ \left[\frac{1}{2}A_2B_1c_1 + \frac{1}{4}A_1B_2c_1^2 + \frac{A_0B_2}{2} c_1 \left(c_2 - \frac{c_1^2}{2} \right) + A_1B_1 \left(c_2 - \frac{c_1^2}{2} \right) \right]$$

$$+ \frac{A_0B_1}{2} \left(c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + \frac{A_0B_3}{8} c_1^3 \Big] z^3 + \dots \tag{14}$$

Considering the LHS of Eq. (13) implies that

$$\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\alpha - 1 = (1 + \alpha)a_2z$$

$$+ \left[(2 + \alpha)a_3 + \frac{(\alpha - 1)(\alpha + 2)}{2}a_2^2 \right] z^2$$

$$+ \left[(3 + \alpha)a_4 + \alpha(\alpha - 1) + 3(\alpha - 1)a_2a_3 \times \right.$$

$$\left. \left(\frac{\alpha(\alpha - 1)(\alpha - 2)}{6} + \frac{\alpha(\alpha - 1)}{2} - \alpha + 1 \right) a_2^3 \right] z^3 + \dots \tag{15}$$

Equating Eqs. (14) and (15) yields

$$a_2 = \frac{A_0B_1c_1}{2(1 + \alpha)} \implies |a_2| \leq \frac{|A_0|B_1|c_1|}{2(1 + \alpha)} \tag{16}$$

and the application of Lemma 2.1 in Eq. (16) gives the estimate in Eq. (11). Also note that

$$a_3 = \frac{A_1B_1c_1 + A_0B_1 \left(c_2 - \frac{c_1^2}{2} \right) + \frac{A_0B_2}{2} c_1^2 - \frac{(\alpha-1)(\alpha+2)A_0^2B_1^2}{8(1+\alpha)^2} c_1^2}{2(2 + \alpha)} \tag{17}$$

which implies that

$$|a_3| \leq \frac{|A_1|B_1|c_1| + |A_0|B_1 \left| c_2 - \frac{c_1^2}{2} \right| + \frac{|A_0||B_2|}{2} |c_1^2| + \frac{|\alpha-1|(\alpha+2)|A_0^2|B_1^2}{8(1+\alpha)^2} |c_1^2|}{2(2 + \alpha)} \tag{18}$$

and the application of Lemmas 2.1 and 2.2 in Eq. (18) gives the estimate in Eq. (12). \square

Theorem 3.5. *If $f \in \mathbb{Q}_q^\alpha(m)$ is given by Eq. (1), then*

$$\nabla(\gamma, f) = |a_3 - \gamma a_2^2| \leq \begin{cases} \frac{|A_0|B_1}{2(2+\alpha)}(-\psi) + \frac{|A_1|B_1}{(2+\alpha)} & \text{for } \gamma \leq \tau_1 \\ \frac{|A_0|B_1}{(2+\alpha)} + \frac{|A_1|B_1}{(2+\alpha)} & \text{for } \tau_1 \leq \gamma \leq \tau_2 \\ \frac{|A_0|B_1}{2(2+\alpha)}\psi + \frac{|A_1|B_1}{(2+\alpha)} & \text{for } \gamma \geq \tau_2 \end{cases},$$

where $\gamma \in \mathbb{R}$,

$$\psi = \frac{(\alpha - 1)(\alpha + 2)A_0B_1^2 + 4\gamma A_0B_1^2(2 + \alpha) - 4(1 + \alpha)^2B_2}{2B_1(1 + \alpha)^2},$$

$$\tau_1 = \left(\frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0B_1}{4(1 + \alpha)^2} - 1 \right) \frac{(1 + \alpha)^2}{A_0B_1(2 + \alpha)},$$

and

$$\tau_2 = \left(\frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0B_1}{4(1 + \alpha)^2} + 1 \right) \frac{(1 + \alpha)^2}{A_0B_1(2 + \alpha)}.$$

Proof. Putting Eqs. (16) and (17) into Eq. (7) means that

$$a_3 - \gamma a_2^2 = \left(\frac{A_1B_1}{2(2 + \alpha)} c_1 + \frac{A_0B_1}{2(2 + \alpha)} \left(c_2 - \frac{c_1^2}{2} \right) \right.$$

$$\left. + \frac{A_0B_2}{4(2 + \alpha)} c_1^2 - \frac{(\alpha - 1)(\alpha + 2)A_0^2B_1^2}{16(1 + \alpha)^2(2 + \alpha)} c_1^2 \right) - \gamma \left(\frac{A_0B_1}{2(1 + \alpha)} c_1 \right)^2$$

where some simplifications lead to

$$a_3 - \gamma a_2^2 = \frac{A_0 B_1}{2(2 + \alpha)} \left\{ c_2 - \left(1 - \frac{B_2}{B_1} + \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + \frac{\gamma A_0 B_1 (2 + \alpha)}{(1 + \alpha)^2} \right) \frac{c_1^2}{2} \right\} + \frac{A_1 B_1}{2(2 + \alpha)} c_1$$

so that

$$|a_3 - \gamma a_2^2| \leq \frac{|A_0 B_1|}{2(2 + \alpha)} \left| c_2 - \lambda \frac{c_1^2}{2} \right| + \frac{|A_1 B_1|}{2(2 + \alpha)} |c_1| \quad (19)$$

for

$$\lambda = 1 - \frac{B_2}{B_1} + \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + \frac{\gamma A_0 B_1 (2 + \alpha)}{(1 + \alpha)^2}. \quad (20)$$

Using Lemma 2.2 in Eq. (19) implies that for $\lambda \leq 0$,

$$\begin{aligned} \left| c_2 - \lambda \frac{c_1^2}{2} \right| &\leq 2 \left[1 - \left(1 - \frac{B_2}{B_1} + \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + \frac{\gamma A_0 B_1 (2 + \alpha)}{(1 + \alpha)^2} \right) \right] \\ &= - \left(\frac{(\alpha - 1)(\alpha + 2)A_0 B_1^2 + 4\gamma A_0 B_1^2 (2 + \alpha) - 4(1 + \alpha)^2 B_2}{2B_1(1 + \alpha)^2} \right) \end{aligned} \quad (21)$$

and from Eq. (20) we get

$$\gamma \leq \left(\frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} - 1 \right) \frac{(1 + \alpha)^2}{A_0 B_1 (2 + \alpha)}. \quad (22)$$

Secondly, for $0 \leq \lambda \leq 2$, we get from Eq. (19) that

$$\left| c_2 - \lambda \frac{c_1^2}{2} \right| \leq 2, \quad (23)$$

and from Eq. (20) we get

$$\begin{aligned} \left(\frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} - 1 \right) \frac{(1 + \alpha)^2}{A_0 B_1 (2 + \alpha)} &\leq \gamma \leq \\ \left(\frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + 1 \right) \frac{(1 + \alpha)^2}{A_0 B_1 (2 + \alpha)}. \end{aligned} \quad (24)$$

Thirdly, for $\lambda \geq 2$, we get from Eq. (19) that

$$\begin{aligned} \left| c_2 - \lambda \frac{c_1^2}{2} \right| &\leq 2 \left[\left(1 - \frac{B_2}{B_1} + \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + \frac{\gamma A_0 B_1 (2 + \alpha)}{(1 + \alpha)^2} \right) \right. \\ &\left. - 1 \right] = \frac{(\alpha - 1)(\alpha + 2)A_0 B_1^2 + 4\gamma A_0 B_1^2 (2 + \alpha) - 4(1 + \alpha)^2 B_2}{2B_1(1 + \alpha)^2}, \end{aligned} \quad (25)$$

and from Eq. (20) we get

$$\gamma \geq \left(\frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + 1 \right) \frac{(1 + \alpha)^2}{A_0 B_1 (2 + \alpha)}. \quad (26)$$

In summary, using Lemma 2.1 and putting Eqs. (21) – (26) into Eq. (19) yield the desired result in the theorem. \square

Theorem 3.6. If $f \in \mathbb{Q}_q^\alpha(m)$ is given by Eq. (1), then

$$\nabla(\gamma, f) = |a_3 - \gamma a_2^2| \leq \frac{|A_0 B_1|}{(2 + \alpha)} \max\{1, \chi\} + \frac{|A_1 B_1|}{(2 + \alpha)},$$

where $\gamma \in \mathbb{C}$ and

$$\chi = \left| \frac{B_2}{B_1} - \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} - \frac{\gamma A_0 B_1 (2 + \alpha)}{(1 + \alpha)^2} \right|.$$

Proof. For $\gamma \in \mathbb{C}$ in Eq. (19) and using Lemma 2.2 implies that

$$1 - \lambda = 1 - \left(1 - \frac{B_2}{B_1} + \frac{(\alpha - 1)(\alpha + 2)A_0 B_1}{4(1 + \alpha)^2} + \frac{\gamma A_0 B_1 (2 + \alpha)}{(1 + \alpha)^2} \right).$$

Some simplifications and the use of Lemma 2.1 yield the desired result. \square

Theorem 3.7. If $f \in \mathbb{Q}_q^\alpha(m)$ is given by Eq. (1), then

$$|\Delta_{2,1}(f)| = |a_3 - a_2^2| \leq \frac{B_1(|A_0| + |A_1|)}{2 + \alpha}.$$

Proof. Letting $\gamma = 1$ in Theorem 3.5 yields the desired result. \square

4. CONCLUSION

The investigations in this paper were on a subclass of analytic-univalent functions defined in the unit disk $\Omega := \{z \mid z \in \mathbb{C}, |z| < 1\}$ herein denoted by $\mathbb{Q}_q^\alpha(m)$. This new class generalized some well-known subclasses of analytic-univalent functions which included the classes of starlike functions, Yamaguchi functions and Ma-Minda functions. The definition of the class involved the use of the principles of Taylor's series and quasi-subordination. The quasi-subordination is known to generalize the principles of subordination and majorization. The investigated properties of the function $f \in \mathbb{Q}_q^\alpha(m)$ however included, the estimates for some coefficient bounds, the solution to the Fekete-Szegő problem and the estimate for a Hankel determinant. Some particular fields of application include conformal mappings, special functions, physics and engineering designs.

References

- [1] P. L. Duren, *Univalent functions*, Springer-Verlag Inc., New York, USA, 1983. <https://link.springer.com/book/9780387907956>.
- [2] A. O. Lasode, A. O. Ajiboye & R. O. Ayinla, "Some coefficient problems of a class of close-to-star functions of type α defined by means of a generalized differential operator", *International Journal of Nonlinear Analysis and Applications* **14** (2023) 1. <https://doi.org/10.22075/ijnaa.2022.26979.3466>.
- [3] D. K. Thomas, N. Tuneski & A. Vasudevarao, *Univalent functions: A primer*, Walter de Gruyter Inc., Berlin, Germany, 2018. <https://doi.org/10.1515/9783110560961-001>.
- [4] P. L. Duren, "Coefficients of univalent functions", *Bulletin of the American Mathematical Society* **83** (1977) 5. <https://doi.org/10.1090/S0002-9904-1977-14324-3>.
- [5] D. L. Branges, "A proof of the Bieberbach conjecture", *Acta Mathematica* **154** (1985) 1. <https://doi.org/10.1007/BF02392821>.
- [6] R. O. Ayinla, "Estimates of Fekete-Szegő functional of a subclass of analytic and bi-univalent functions by means of Chebyshev polynomials", *Journal of Progressive Research in Mathematics* **18** (2021) 1. <http://www.scitecresearch.com/journals/index.php/jprm/article/view/1979/1406>.
- [7] R. O. Ayinla & A. O. Lasode, "On a certain class of bi-univalent functions in connection with Gegenbauer polynomials, *Pan-American Journal of Mathematics* **3** (2024) 5. <https://doi.org/10.28919/cpr-pajm/3-5>.

- [8] R. O. Ayinla & T. O. Opoola, "The Fekete Szegő functional and second Hankel determinant for a certain subclass of analytic functions", *Applied Mathematics* **10** (2019) 12. <https://doi.org/10.4236/am.2019.1012074>.
- [9] R. O. Ayinla & T. O. Opoola, "Initial coefficient bounds and second Hankel determinant for a certain class of bi-univalent functions using Chebyshev polynomials", *Gulf Journal of Mathematics* **14** (2023) 1. <https://doi.org/10.56947/gjom.v14i1.1092>.
- [10] A. O. Lasode, "Estimates for a generalized class of analytic and bi-univalent functions involving two q -operators", *Earthline Journal of Mathematical Sciences* **10** (2022) 2. <https://doi.org/10.34198/ejms.10222.211225>.
- [11] A. O. Lasode & T. O. Opoola, "Coefficient problems of a class of q -starlike functions associated with q -analogue of Al-Oboudi-Al-Qahtani integral operator and nephroid domain", *Journal of Classical Analysis* **20** (2022) 1. <https://doi.org/10.7153/jca-2022-20-04>.
- [12] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions", *Proceedings of the London Mathematical Society* **41** (1966) 1. <https://doi.org/10.1112/jlms/s1-41.1.111>.
- [13] M. S. Robertson, "Quasi-subordination and coefficients conjectures", *Bulletin of American Mathematical Society* **76** (1970). <https://www.ams.org/journals/bull/1970-76-01/S0002-9904-1970-12356-4/S0002-9904-1970-12356-4.pdf>.
- [14] T. H. MacGregor, "Majorization by univalent functions", *Duke Mathematical Journal* **34** (1967) 1. <https://doi.org/10.1215/S0012-7094-67-03411-4>.
- [15] S. O. Olatunji & S. Altinkaya, "Generalized distribution associated with quasi-subordination in terms of error function and Bell numbers", *Jordan Journal of Mathematics and Statistics* **14** (2021) 1. <https://doi.org/10.47013/14.1.5>.
- [16] E. A. Oyekan, A. O. Lasode & T. G. Shaba, "On a set of generalized Janowski-type starlike functions connected with Mathieu-type series and Opoola differential operator", *Pan-American Journal of Mathematics* **2** (2023) 11. <https://doi.org/10.28919/cpr-pajm/2-11>.
- [17] E. A. Oyekan, S. R. Swamy, P. O. Adepoju & T. A. Olatunji, "Quasi-convolution properties of a new family of close-to-convex functions involving a q - p -Opoola differential operator", *International Journal of Mathematics Trends and Technology* **69** (2023) 5. <https://doi.org/10.14445/22315373/IJMTT-V69I5P506>.
- [18] W. C. Ma & D. Minda, "A unified treatment of some special classes of univalent functions", presented at the International Conference on Complex Analysis, Nankai Institute of Mathematics, Nankai University, Tianjin, China, 1994. https://www.researchgate.net/profile/C-Minda/publication/245129813_A_unified_treatment_of_some_special_classes_of_functions/links/543693bf0cf2bf1f1f2be1b2/A-unified-treatment-of-some-special-classes-of-functions.pdf
- [19] A. O. Lasode & T. O. Opoola, "Hankel determinant of a subclass of analytic and bi-univalent functions defined by means of subordination and q -differentiation", *International Journal of Nonlinear Analysis and Applications* **13** (2022) 2. <https://doi.org/10.22075/IJNAA.2022.24577.2775>.
- [20] K. O. Babalola & T. O. Opoola, "On the coefficients of a certain class of analytic functions", in *Advances in Inequalities for Series*, S. S. Dragomir & A. Sofo (Eds.), Nova Science Publishers Inc., Hauppauge, New York, USA, 2008, pp. 1–13. <https://www.amazon.com/Advances-Inequalities-M-Darus/dp/1600219209>.
- [21] K. Yamaguchi, "On functions satisfying $\Re(f(z)/z) > 0$ ", *Proceedings of the American Mathematical Society* **17** (1966) 3. <https://doi.org/10.1090/S0002-9939-1966-0192041-7>.