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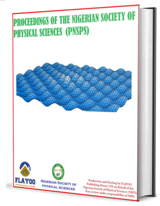

Application of shifted Vieta-Lucas polynomials for the numerical treatment of Volterra-integro differential equations

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ABSTRACT

In this study, the numerical solution of the Volterra-integro differential equations was obtained by applying the variational iteration strategy with the shifted Vieta-Lucas polynomials. The proposed method builds the shifted Vieta-Lucas polynomials for the Volterra-integro differential equation which are then used as basis functions for the approximation. Numerical examples were given to establish the effectiveness and dependability of the recommended approach. Calculations were performed using Maple 2022 software.

Keywords: Variational iteration method, Volterra integro differential equation, shifted Vieta-Lucas polynomials.

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1. INTRODUCTION

Volterra integro-differential equation (IDE) is widely used in many fields of science and engineering and in particular, Fluid dynamics and its applications. The Volterra IDE incorporates models for problems involving ordinary differential equations and partial differential equations with boundary and initial conditions which made it of interest and worth while to study. Different numerical techniques have previously been used to study the Volterra IDE, numerically, but the work of Wazwaz [1] is of the major interest in this study. The author investigated these problems using Adomian decomposition method (ADM), variational iteration method (VIM), power series method, homotopy perturbation method (HPM) and the modified Adomian decomposition method (MADM). Various collocation techniques were also used by Refs. [2–6]. Another method to seek the numerical solution of Volterra IDE is the linear multistep method [7, 8], Collocation approach was also used for solution of Fredholm-Volterra fractional order of integro-differential equations [9], Bernoulli matrix method was used to solve nonlinear Fredholm integro-differential equations [10], differential transform method [11], pseudospectral method [12], Mellin transform method [13], wavelet-based method [14], Chebyshev computational approach [15] and shifted Vieta-Lucas polynomials [16, 17].

Ref. [6] outlined an effective numerical technique for resolving Volterra IDEs utilizing power series as a basis function, and obtained results for $N = 4, 5$ and 7 in the problem under consider-

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ation. In the class of IDEs under consideration, he substituted the power series approximation. The results obtained for a few numerical examples demonstrate the effectiveness of the suggested approach.

Motivated by the aforementioned studies, we are concerned with the Volterra differential equation of the form:

$$Z^n(\alpha) = f(\alpha) + \int_0^\alpha K(\alpha, t) Z(t)\, dt, \tag{1}$$

subject to the initial condition

$$Z^n(\alpha) = a_n, \quad n = 0, 1, 2 \ldots N,$$

where $a \leq \tau \leq b$, $Z^n(\alpha)$ is the unknown function, $K(\alpha, t)$ is the Voltera integral kernel function and $f(\alpha)$ the known function.

In order to solve the Volterra integro-differential equation numerically, the variational iteration method is paired with shifted Vieta-Lucas polynomials to achieve a converging series solution.

2. THE STANDARD VARIATIONAL ITERATION METHOD WITH VOLTERRA- INTEGRO DIFFERENTIAL EQUATION

The core notion of the procedure is illustrated by the Volterra-integro differential equation:

$$Z^n(\alpha) = f(\alpha) + \int_0^\alpha K(\alpha, t) Z(t)\, dt, \tag{2}$$

where $Z^n(\alpha)$ is the unknown function, $K(\alpha, t)$ is the Volterra integral kernel function and $f(\alpha)$ the known function. According to variational iteration algorithm, we can construct a correction functional as follows:

$$Z_{m+1}(\alpha) = Z_m(\alpha) + \int_0^\alpha \lambda(s) \left(\frac{d^n Z}{ds^n} - f(s) - \int_0^s K(s, t) Z(t)\, dt \right) ds, \tag{3}$$

where $\lambda(t)$ is a Lagrange multiplier that can be best determined using a VIA. The m^{th} approximation is indicated by the subscript m, and $\widehat{w}_m$ is regarded as a restricted variation, i.e., $\widehat{Z}_m = 0$. The relation (3) is called a correction functional. Due to the precise identification of the Lagrange multiplier, both linear and nonlinear problems can be solved in a single iteration step. In this method, we must select the Lagrange multiplier $\lambda(t)$ ideally, thus it will be simple to construct each subsequent estimate of solution w by applying both the Lagrange multiplier and the Z_0, and the solution is given by:

$$\lim_{m-\infty} Z_m = Z\,.$$

The Lagrange multiplier, which can be described as follows, is also crucial in determining how the problem will be solved.

$$(-1)^m \frac{1}{(m-1)!} (t - \alpha)^{m-1}\,.$$

3. VIETA-LUCAS POLYNOMIALS

The Vieta-Lucas polynomials are orthogonal polynomials with $|\alpha| < 2$, defined as: $VL_n(\alpha) = 2\cos(n\theta)$, where $\theta = \cos^{-1}\left(\frac{\alpha}{2}\right)$, $\theta \in [0, \pi]$. $VL_n(\alpha)$ is also obtained through the following explicit power series formula:

$$VL_n(\alpha) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} (-1)^i \frac{n\Gamma(n-i)}{\Gamma(i+1)\Gamma(n+1-2i)} \tau^{n-2i, n=\{2,3,\ldots\}},$$

where $\lceil \frac{n}{2} \rceil$ is called the ceiling function.

The polynomial $VL_n(\alpha)$ can be generated by the following iterative formula: $VL_n(\alpha) = \tau VL_{n-1}(\alpha) - VL_{n-2}(\alpha)$, $n = 2, 3, \ldots,$ with $VL_0(\alpha) = 2$ and $VL_1(\alpha) = \alpha$. Hence, the first few Vieta-Lucas polynomials are given as:

$$VL_0(\alpha) = 2, VL_1(\alpha) = \alpha, \; VL_2(\alpha) = \alpha^2 - 2, \ldots. \tag{4}$$

4. SHIFTED VIETA-LUCAS POLYNOMIALS

The shifted Vieta-Lucas polynomials of degree n on [0, 1] can be derived from $VL_n(\alpha)$ as follows:

$$VL_n^*(\alpha) = VL_n(4\alpha - 2) = VL_{2n}\left(2\sqrt{\alpha}\right).$$

$VL_n{}^*(\alpha)$ is also obtained through the following explicit power series formula:

$$VL_n{}^*(\alpha) = 2n \sum_{i=0}^{n} (-1)^i \frac{4^{n-i}\Gamma(2n-i)}{\Gamma(i+1)\Gamma(2n-2i+1)} \tau^{n-i, n=\{2,3,\ldots\}}.$$

The polynomial $VL_n{}^*(\alpha)$ can be generated by the following iterative formula: $VL_{n+1}^*(\alpha) = (4\alpha - 2) VL_n{}^*(\alpha) - VL_{n-1}{}^*(\alpha)$, $n = 1, 2, \ldots,$ with $VL_0{}^*(\alpha) = 2$ and $VL_1{}^*(\alpha) = 4\alpha - 2$. Hence, the first few shifted Vieta-Lucas polynomials are given as:

$$VL_0^*(\alpha) = 2, \; VL_1^*(\alpha) = 4\alpha - 2, \; VL_2^*(\alpha) = 16\alpha^2 - 16\alpha + 2. \tag{5}$$

5. VARIATIONAL ITERATION ALGORITHM FOR VOLTERA INTEGRO-DIFFERENTIAL EQUATION MIXED WITH SHIFTED VIETA-LUCAS POLYNOMIALS

Using equations (2) and (3), we assume an approximate solution of the form:

$$Z_{m,N-1}(\alpha) = \sum_{m=0}^{N-1} \xi_{m,N-1} VL_{m,N-1}^*(\alpha),$$

where $VL_{m,N-1}^*(\alpha)$ are shifted Vieta-Lucas polynomials, $\delta_{m,N-1}$ are constants to be determined, and N the degree of approximant. Hence we obtain the following iterative method

$$Z_{m+1}(\alpha) = \sum_{m=0}^{N-1} \delta_{m,N-1} VL_{m,N-1}^*(\alpha) + \int_0^\alpha \lambda(s) \left(\frac{d^n Z}{ds^n} \left(\sum_{m=0}^{N-1} \delta_{m,N-1} VL_{m,N-1}^* \tau \right) - f(s) - \int_0^s K(s, t) \left(\sum_{m=0}^{N-1} \delta_{m,N-1} VL_{m,N-1}{}^*(t) \right) dt \right) ds. \tag{6}$$

6. CONVERGENCE OF THE METHOD

The Banach's theorem concerning the variational iteration algorithm's convergence utilizing shifted Vieta-Lucas polynomials will be discussed in this part. The method turns the given differential equation into a sequence of function recurrences. It is presumed that the given differential equation has a solution at the limit of this sequence.

THEOREM 1

Given that $\mathfrak{H}:\mathcal{M}\rightarrow\mathcal{M}$ is a nonlinear mapping and $\mathcal{M}$ be a Banach space, it is assumed that

$$\left\|\mathfrak{H}[Z]-\mathfrak{H}[\overline{Z}]\right\|\leq\zeta\left\|Z-\overline{Z}\right\|,\forall Z,\overline{Z}\in\mathcal{M}:\zeta<1. \tag{7}$$

Then $\mathfrak{H}$ has a unique fixed point. Hence, the sequence

$$Z_{m+1}=\mathfrak{H}[Z_m], \tag{8}$$

with an arbitrary choice of $Z_0\in\mathcal{M}$ converges to the fixed point of $\mathfrak{H}$ and we have that

$$\begin{aligned}&\left\|Z_p-Z_1\right\|\leq\left\|Z_p-Z_{p-1}\right\|+\cdots+\left\|Z_{q+1}-Z_q\right\|\\&\left\|\mathfrak{H}\left(Z_{p-1}\right)-\mathfrak{H}\left(Z_{p-2}\right)\right\|+\cdots+\left\|\mathfrak{H}\left(Z_q\right)-\mathfrak{H}\left(Z_{q-1}\right)\right\|\\&\leq\zeta\left\|Z_{p-1}-Z_{p-2}\right\|+\cdots+\zeta\left\|Z_q-Z_{q-1}\right\|\\&\leq\left(\zeta^{p-2}+\zeta^{p-3}+\cdots+\zeta^{q-1}\right)\|Z_1-Z_0\|\leq\frac{\zeta^{q-1}}{1-\zeta}\|Z_1-Z_0\|,\end{aligned} \tag{9}$$

where $\zeta<1$, with the assumption that $p>q\geq1$. This gives $\left\|Z_p-Z_q\right\|\longrightarrow0$ as $p,q\longrightarrow\infty$ and hence the sequence $\{Z_p: p=1\ldots\infty\}$ is Cauchy. The sequence converges to a fixed point since M is a Banach space and therefore convergent. According to Theorem 1, we obtain that

$$\mathfrak{H}[Z]=Z_{m,N-1}(\alpha)+\int_0^\alpha\lambda(t)\left(LZ_m(t)+N\widehat{Z_m(t)}-g(t)\right)dt, \tag{10}$$

$$\mathfrak{H}[Z]=\sum_{m=0}^{N-1}\delta_{m,N-1}VL^*_{m,N-1}(\alpha)+\int_0^\alpha\lambda(s)\left(\frac{d^nZ}{ds^n}-f(s)-\int_0^sK(s,t)Z(t)\,dt\right)ds, \tag{11}$$

and this is a sufficient condition for the convergence of the VIM using shifted Vieta-Lucas polynomials, which is strictly a contraction on $\mathfrak{H}$. In the sequel, the sequence (8) converges to a fixed point of $\mathfrak{H}$ which is also a solution of equation (2).

7. NUMERICAL APPLICATIONS

In this section, we apply the proposed methodology to three problems. Furthermore, numerical results show the accuracy and efficiency of the proposed approach.

Example 1: Consider the second order Volterra IDE [1]:

$$Z''(\alpha)=1+\alpha+\int_0^\alpha(\alpha-t)Z(t)\,dt, \tag{12}$$

with initial conditions

$$Z(0)=1,Z'(0)=1. \tag{13}$$

The exact solution for the problem is $Z(\alpha)=\exp(\alpha)$.

The initial value problem is corrected with the following functional:

$$Z_{m+1}=Z_m(\alpha)+\int_0^\alpha\lambda(s)\left(\frac{d^2Z}{ds^2}-1-s-\int_0^s(s-t).Z(t)\,dt\right)ds,$$

where $\lambda(t)=(t-\alpha)$ is the Lagrange multiplier.

Table 1. Comparison of numerical findings of example 1.

α	Exact solution	Approximate solution	Absolute error by the proposed method
0.0	1.000000000	1.000000000	0.000000E-00
0.2	1.221402758	1.221402667	9.100000E-08
0.4	1.491824698	1.491818667	6.031000E-07
0.6	1.822118800	1.822048000	7.080000E-06
0.8	2.225540928	2.225130667	4.102610E-05
1.0	2.718281828	2.716666667	1.615161E-04

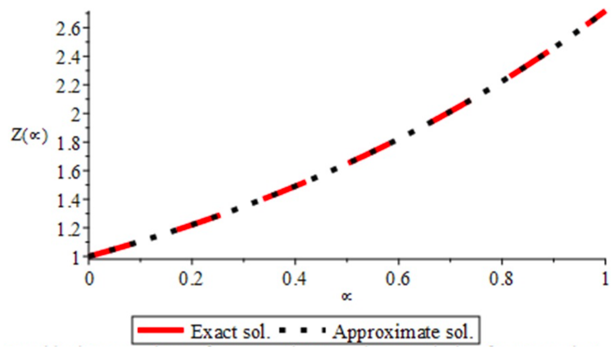

Figure 1. Comparison of exact and approximate solutions of example 1.

Using the VIM coupled with the SVLPs, we assume an approximate solution of this type.

$$Z_{m,1}(\alpha)=\sum_{m=0}^{1}\delta_{m,1}V_L{}^*{}_{m,1}(\alpha).$$

Consequently, we obtain the iterative formula as follows:

$$Z_{m+1,N-1}=\sum_{m=0}^{1}\delta_{m,1}V^*_{Lm,1}(\alpha)+\int_0^\alpha\lambda(s)\left(\frac{d^2}{ds^2}\left(\sum_{m=0}^{1}\delta_{m,1}V^*_{Lm,1}(s)\right)-1-s-\int_0^s(s-t)\left(\sum_{m=0}^{1}\delta_{m,1}V^*_{Lm,1}(t)\right)dt\right)ds,$$

$$\begin{aligned}Z_{m+1,N-1}(\alpha)&=\delta_{0,1}V_L{}^*{}_{0,1}(\alpha)+\delta_{1,1}V_L{}^*{}_{1,1}(\alpha)\\&+\int_0^\alpha(s-\alpha)\left(\frac{d^2}{dt^2}\left(\delta_{0,1}V_L{}^*{}_{0,1}(s)+\delta_{1,1}V_L{}^*{}_{1,1}(s)\right)-1-s\right.\\&\left.-\int_0^s(s-t)\left(\delta_{0,1}V_L{}^*{}_{0,1}(t)+\delta_{1,1}V_L{}^*{}_{1,1}(t)\right)dt\right)ds.\end{aligned}$$

The initial conditions in equation (13) were used to determine the values of the unknown constants $\delta_{0,1}=0.750000000$, $\delta_{1,1}=0.25000000$. Therefore, the series solution is provided as

$$Z(\alpha)=1+\alpha+\frac{1}{2}\alpha^2+\frac{1}{6}\alpha^3+\frac{1}{24}\alpha^4+\frac{1}{120}\alpha^5+(O)^6.$$

The numerical results are shown in Table 1 and Figure 1.

Example 2 Consider the third order Volterra IDE [1]:

$$Z'''(\alpha)=-1+\alpha-\int_0^\alpha(\alpha-t)Z(t)\,dt, \tag{14}$$

Table 2. Comparison of numerical findings of example 2.

α	Exact solution	Approximate solution	Absolute error by the proposed method
0.0	1.0000000000	1.0000000000	0.0000000E-00
0.2	0.8187307531	0.8187307530	1.0000000E-10
0.4	0.6703200460	0.6703200305	1.5500000E-08
0.6	0.5488116361	0.5488112457	3.9040000E-07
0.8	0.4493289641	0.4493251454	3.8187000E-07
1.0	0.3678794412	0.3678571428	2.2298400E-06

with initial conditions

$$Z(0) = 1, Z'(0) = -1, Z''(0) = 1. \tag{15}$$

The exact solution for the problem is $Z(\alpha) = \exp(-\alpha)$.

The initial value problem is corrected with the following functional:

$$Z_{m+1} = Z_m(\alpha) + \int_0^\alpha \lambda(s)\left(\frac{d^3Z}{ds^3} + 1 - s - \int_0^s (s-t).Z(t)\,dt\right)ds,$$

where $\lambda(t) = \frac{(-1)^3(t-\alpha)^2}{3!}$ is the Lagrange multiplier.

Using the VIM and the SVLPs, we assume an approximate solution of this type.

$$Z_{m,2}(\alpha) = \sum_{m=0}^{2} \delta_{m,2} V_L{}^*{}_{m,2}(\alpha).$$

Consequently, we obtain the iterative formula as follows:

$$\begin{aligned} Z_{m+1,N-1} &= \sum_{m=0}^{2} \delta_{m,2} V_L{}^*{}_{m,2}(\alpha) \\ &+ \int_0^\alpha \lambda(t)\left(\frac{d^3}{ds^3}\left(\sum_{m=0}^{2} \delta_{m,2} V_L{}^*{}_{m,2}(s)\right) + 1 - s \right.\\ &\left. + \int_0^s (s-t).\left(\sum_{m=0}^{2} \delta_{m,2} V_L{}^*{}_{m,2}(t)\right)dt\right)ds, \end{aligned}$$

$$\begin{aligned} &Z_{m+1,N-1}(\alpha) = \delta_{0,2} V_L{}^*{}_{0,2}(\alpha) + \delta_{1,2} V_L{}^*{}_{1,2}(\alpha) + \delta_{2,2} V_L{}^*{}_{2,2}(\alpha) \\ &+ \int_0^\alpha \frac{(-1)^3(s-\alpha)^2}{3!}\left(\frac{d^3}{ds^3}\left(\delta_{0,2} V^*_{L0,2}(s) + \delta_{1,2} V^*_{L1,2}(s)\right.\right. \\ &\left. + \delta_{2,2} V_L{}^*{}_{2,2}(s)\right) + 1 - s + \int_0^s (s-t)\left(\delta_{0,2} V_L{}^*{}_{0,2}(t)\right. \\ &\left.\left. + \delta_{1,2} V_L{}^*{}_{1,2}(t) + \delta_{2,2} V_L{}^*{}_{2,2}(t)\right)dt\right)ds. \end{aligned}$$

The initial conditions in equation (15) were used to determine the values of the unknown constants. $\delta_{0,2} = 0.3437500000$, $\delta_{1,2} = -0.1250000000$, $\delta_{2,2} = 0.0312500000$. Therefore, the series solution is provided as

$$Z(\alpha) = 1 - \alpha + \frac{1}{2}\alpha^2 - \frac{1}{6}\alpha^3 + \frac{1}{24}\alpha^4 - \frac{1}{120}\alpha^5 + (O)^6.$$

The numerical results are shown in Table 2 and Figure 2.

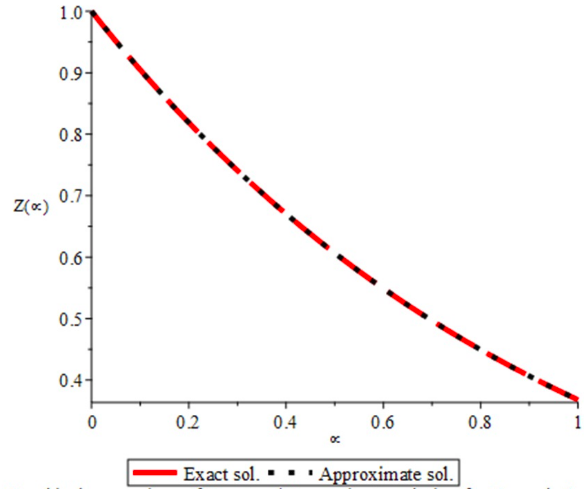

Figure 2. Comparison of exact and approximate solutions of example 2.

Example 3: Consider the fourth order Volterra IDE [1]:

$$Z^{(iv)}(\alpha) = -1 + \alpha - \int_0^\alpha (\alpha - t) Z(t)\,dt, \tag{16}$$

with initial conditions

$$Z(0) = -1,\ Z'(0) = 1,\ Z''(0) = 1,\ Z'''(0) = -1. \tag{17}$$

The exact solution for the problem is $Z(\alpha) = \sin\alpha - \cos\alpha$.

The initial value problem is corrected with the following functional:

$$Z_{m+1} = Z_m(\alpha) + \int_0^\alpha \lambda(s)\left(\frac{d^4 w}{ds^4} + 1 - s + \int_0^s (s-t)\,w(t)\,dt\right)ds,$$

where $\lambda(t) = \frac{(-1)^4(t-\alpha)^3}{4!}$ is the Lagrange multiplier.

Using the SVLPs and the VIM, we assume an approximate solution of the type:

$$Z_{m,3}(\alpha) = \sum_{m=0}^{3} \delta_{m,3} V_L{}^*{}_{m,3}(\alpha).$$

Consequently, we obtain the iterative formula as follows:

$$\begin{aligned} Z_{m+1,N-1}(\alpha) &= \sum_{m=0}^{3} \delta_{m,3} V_L{}^*{}_{m,3}(\alpha) \\ &+ \int_0^\alpha \lambda(s)\left(\frac{d^4}{ds^4}\left(\sum_{m=0}^{3} \delta_{m,3} V_L{}^*{}_{m,3}(s)\right) + 1 - s \right.\\ &\left. + \int_0^s (s-t)\left(\sum_{m=0}^{3} \delta_{m,3} V_L{}^*{}_{m,3}(t)\right)dt\right)ds, \end{aligned}$$

$$\begin{aligned} &Z_{m+1,N-1}(\alpha) = \delta_{0,3} V_L{}^*{}_{0,3}(\alpha) + \delta_{1,3} V_L{}^*{}_{1,3}(\alpha) + \delta_{2,3} V_L{}^*{}_{2,3}(\alpha) \\ &+ \delta_{3,3} V_L{}^*{}_{3,3}(\alpha) + \int_0^\alpha \frac{(-1)^4(s-\alpha)^3}{4!}\left(\frac{d^4}{ds^4}\left(\delta_{0,3} V_L{}^*{}_{0,3}(s)\right.\right. \\ &\left. + \delta_{1,3} V_L{}^*{}_{1,3}(s) + \delta_{2,3} V_L{}^*{}_{2,3}(s) + \delta_{3,3} V_L{}^*{}_{3,3}(s)\right) + 1 - s \\ &+ \int_0^s (s-t)\left(\delta_{0,3} V_L{}^*{}_{0,3}(t) + \delta_{1,3} V_L{}^*{}_{1,3}(t) + \delta_{2,3} V^*_{L2,3}(t)\right. \\ &\left.\left. + \delta_{3,3} V^*_{L3,3}(t)\right)dt\right)ds. \end{aligned}$$

Table 3. Comparison of numerical findings of example 3.

α	Exact solution	Approximate solution	Absolute error by the proposed method
0.0	- 1.00000000000	- 0.9999999997	3.000000E-10
0.2	- 0.7813972470	- 0.7813972521	5.100000E-09
0.4	- 0.5316426517	- 0.5316433763	7.246000E-07
0.6	- 0.2606931415	- 0.2607060843	1.294280E-05
0.8	0.0206493816	0.0205486583	1.007233E-04
1.0	0.3011686789	0.3006723993	4.962796E-04

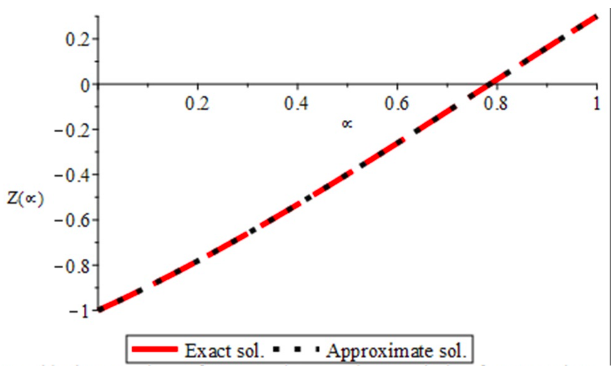

Figure 3. Comparison of exact and approximate solutions of example 3.

The initial conditions in equation (17) were used to determine the values of the unknown constants. $\delta_{0,3} = -0.1822916667$, $\delta_{1,3} = 0.3359375000$, $\delta_{2,3} = 0.0156250000$, $\delta_{3,3} = -0.0022604166667$. Therefore, the series solution is provided as

$$Z(\alpha) = -0.9999999997 + \alpha + \frac{1}{2}\alpha^2 - \frac{1}{6}\alpha^3 - \frac{1}{24}\alpha^4 + \frac{1}{120}\alpha^5 + (O)^6.$$

The numerical results are shown in Table 3 and Figure 3.

8. CONCLUSION

This study explores and successfully applies the VIM with SVLPs to produce numerical solutions for Volterra IDEs. The technique for solving the problem consists of SVLPs and a variational iteration algorithm. For physical problems, this technique yields series solutions that are more realistic and converge quite quickly. Above all, the numerical results demonstrated that the current methodology is an effective mathematical strategy for solving the class of problems that are being studied. This method can be applied to other forms of equations with wider applications. The work can also find application in quantum physics.

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References

[1] A. Wazwaz, *Linear and nonlinear integral equations: methods and applications*, Springer Berlin, Heidelberg, 2011. https://link.springer.com/book/10.1007/978-3-642-21449-3.

[2] A. O. Adesanya, Y. A. Yahaya, B. Ahmed & R. O. Onsachi, "Numerical solution of linear integral and integro-differential equations using boubakar collocation method", International Journal of Mathematical Analysis and Optimization: Theory and Application **2** (2019) 592. http://ijmso.unilag.edu.ng/article/view/566.

[3] D. A. Gegele, O. P. Evans & D. Akoh, "Numerical solution of higher order linear Fredholm integro-differential equations", American Journal of Engineering Research **3** (2014) 243. https://www.ajer.org/papers/v3(8)/Y03802430247.pdf.

[4] S. Nemati, P. Lima & Y. Ordokhani, "Numerical method for the mixed Volterra-Fredholm integral equations using hybrid Legendre function", Conference Application of Mathematics (2015) 184. https://am2015.math.cas.cz/proceedings/contributions/nemati.pdf.

[5] A. O. Agbolade & T. A. Anake, "Solution of first order Voltera linear integro differential equations by collocation method", J. Appl. Math **2017** (2017) 1510267. https://doi.org/10.1155/2017/1510267.

[6] G. Ajileye, S. A. Amoo, "Numerical solution to Volterra integro-differential equations using collocation approximation", Mathematics and computational Science **4** (2023) 1. https://doi.org/10.30511/mcs.2023.1978083.1099.

[7] G. Mehdiyeva, V. Ibrahimov & M. Imanova, "On the construction of the multistep methods to solving the initial-value problem for ode and the volterra integro-differential equations", Proceedings of the International Conference on Indefinite Applied Energy (IAPE 2019). http://dx.doi.org/10.17501.

[8] G. Mehdiyera, M. Imanova & V. Ibrahim, "Solving Volterra integro differential equation by second derivative methods", Appl. Math. Inf. Sci. **9** (2015) 2521. http://dx.doi.org/10.12785/amis/090536.

[9] G. Ajileye, A. A. James, A. M. Ayinde & T. Oyedepo, "Collocation approach for the computational solution of fredholm-volterra fractional order of integro-differential equations", J. Nig. Soc. Phys. Sci. **4** (2022) 834. https://doi.org/10.46481/jnsps.2022.834.

[10] A. H. Bhraway, E. Tohidi & F. Soleymani, "A new Bernoulli matrix method for solving high order linear and nonlinear Fredholm integro-differential equations with piecewise interval", Appl. Math. Comput. **219** (2012) 482. https://doi.org/10.1016/j.amc.2012.06.020.

[11] C. Ercan & T. Kharerah, "Solving a class of Volterra integral system by the differential transform method", Int. J. Nonlinear Sci. **16** (2013) 87. https://www.researchgate.net/publication/330401134_Solving_a_Class_of_Volterra_Integral_Equation_Systems_by_the_Differential_Transform_Method.

[12] M. El-kady & M. Biomy, "Efficient Legendre pseudospectral method for solving integral and integro differential equation", Commom Nonlinear Sci. Numer Simulat **15** (2010) 1724. https://doi.org/10.1016/j.cnsns.2009.07.012.

[13] S. E. Fadugba, "Solution of fractional order equations in the domain of the Mellin transform", Journal of the Nigerian Society of Physical Sciences **1** (2019) 138. https://doi.org/10.46481/jnsps.2019.31.

[14] M. Sohaib & S. Haq. "An efficient wavelet-based method for numerical solution of nonlinear integral and integro-differential equations", Mathematical Methods in the Applied Sciences (2020) 1. https://doi.org/10.1002/mma.6441.

[15] T. Oyedepo, A. A. Ayoade, M. O. Oluwayemi & R. Pandurangan, "Solution of Volterra-Fredholm integro-differential equations using the Chebyshev computational approach", International Conference on Science, Engineering and Business for Sustainable Development Goals (SEB-SDG), Omu-Aran, Nigeria (2023) 1. https://doi.org/10.1109/seb-sdg57117.2023.10124647.

[16] I. J. Otaide, K. O. Ogeh, M. I. Modebei, T. Oyedepo & M. O. Oluwayemi, "Variational iteration algorithm for numerical solutions of sixth and seventh order boundary value problems using shifted Vieta-Lucas polynomials", Scientific African **22** (2023) e01924. https://doi.org/10.1016/j.sciaf.2023.e01924.

[17] G. Ajileye & F. A. Aminu, "Approximate Solution to First-Order Integro-differential Equations Using Polynomial Collocation Approach, J Appl Computat Math. **11** (2022) 486. https://doi.org/10.37421/2168-9679.2022.11.486.