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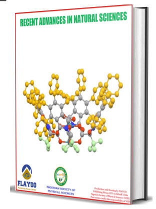

Some numerical significance of the Riemann zeta function

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ABSTRACT

In this paper, the Riemann analytic continuation formula (RACF) is derived from Euler's quadratic equation. A nonlinear function and a polynomial function that were required in the derivation were also obtained. The connections between the roots of Euler's quadratic equation and the Riemann zeta function (RZF) are also presented in this paper. The method of partial summation was applied to the series that was obtained from the transformation of Euler's quadratic equation (EQE). This led to the derivation of the RACF. A general equation for the generation of the zeros of the analytic continuation formula of the Riemann Zeta equation via a polynomial approach was also derived and thus presented in this work. An expression, which was based on a polynomial function and the products of prime numbers, was also obtained. The obtained function thus afforded us an alternative approach to defining the analytic continuation formula of the Riemann zeta equation (ACFR). With the new representation, the Riemann zeta function was shown to be a type of function. We were able to show that the solutions of the RACF are connected to some algebraic functions, and these algebraic functions were shown to be connected to the polynomial and the nonlinear functions. The tables and graphs of the numerical values of the polynomial and the nonlinear function were computed for a generating parameter, k, and shown to be some types of the solutions of some algebraic functions. In conclusion, the RZF was redefined as the product of a derived function, $R(t_n, s)$, and it was shown to be dependent on the obtained polynomial function.

1. INTRODUCTION

Many authors have recently presented some polynomial approaches to proving the Riemann hypothesis. It is interesting to note that Jensen polynomials, Laguerre polynomials, and Jensen polynomials of Laguerre-Pólyaentire functions have been used by authors like [1–9] to establish some possible proofs of the Riemann Hypothesis. Some worked on a general theorem that modeled the used polynomials as Hermite polynomials. In Ref. [9], the authors used fractional calculus to present an approximation to the zeros of the Riemann zeta function. The authors constructed a fractional iterative method to obtain the zeros of functions in which it was possible to avoid expressions that involve hypergeometric functions, Mittag-Leffler functions, or infinite series (cf. Ref. [9]), to mention a few.

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As good as their works were, it was expedient to seek a clearer insight into the nature of these polynomials, knowing that the solutions to the most difficult problems may not necessarily be complex in themselves. It is essential to know that the Riemann zeta function is a function whose solutions are connected to some algebraic functions, and polynomials are also types of solutions of algebraic functions. With these in mind, a polynomial approach was anticipated to prove the Riemann Hypothesis, and the results are presented in this paper.

This study aims to present the links that connect Euler's quadratic equation (EQE) and the analytical continuation formula of the Riemann zeta equation (ACF) by using the generalized polynomial function of Euler's quadratic equation (EQE) and the method of partial summation.

The remaining part of this paper is organized as follows: Section 2 presents the materials and methods. The link between the ACFR and EQE is presented in Section 2.1. The analysis of the derivation of the analytical continuation formula of the Riemann zeta equation (ACFR) from Euler's quadratic equation (EQE) is presented in Section 2.2. Section 3 considers and presents numerical estimates which are of great significance. Section 4 is for the proof of an equivalent equation to the ACFR. In Section 5, the concluding remark is presented.

2. MATERIALS AND METHODS

The derivation of the ACFR from EQE [10] will be obtained from the applications of the following equations and methods:

(i) EQE and the non-conventional expression for -1 in Ref. [11]

(ii) The method of partial summation [10–16]

$$\sum_{pq\leq x}\log p\log q=\sum_{p\leq x}\log p\sum_{q\leq \frac{x}{p}}.\log q \quad (cf.Ref.[15]) \quad (1)$$

(iii)
$$\prod\left(\frac{s}{2}\right)=-\frac{s}{2}\Gamma\left(\frac{s}{2}\right)=\sum_{n\geq 1}^{\infty}\left(\frac{2n+s}{2n}\right). \quad (2)$$

These equations will be used to derive the RAC's derivation from the EQE and to understand some basic components associated with the Riemann Hypothesis. It will be possible to see the links between the roots of the EQE, which is of the form, and the nontrivial zeros of the Riemann zeta function, which is also of the form; $x = 0.5 \pm it$. It is good to note at this point that all the nontrivial zeros of the Riemann zeta Function always have their real parts to be 0.5, which are synonymous with the real part of the roots of the EQE, and all the nontrivial zeros are expected to be within the critical strip. Euler discovered that

$$P(x) = x^2 + x + 41, \quad (3)$$

would always be prime for $1 \leq x \leq 40$. Euler obtained the first few prime numbers from this quadratic Eq. (3) [11]. It can be seen that the roots of (3) are 6.3836i, which have the same real part as those of the nontrivial zeros of the Riemann zeta function. Taking the coefficients of x^2 and x in (3) as as k, and replace 41 with $B(t_n)$ to have:

$$\mu_E(s) = (ks^2 - ks + B(t_n)). \quad (4)$$

Multiplying Eq. (4) by $s + 2n$, whose roots are always $-2n : n = 1, 2, 3, ...$, then we have

$$\zeta_E(s) = (ks^2 - ks + B(t_n))(s + 2n). \quad (5)$$

By the analysis of Eq. (5), the roots of the polynomial were obtained to be the same as the trivial and the nontrivial zeros of the Riemann zeta function, provided that $B(t_n)$ was known [5]. In Ref. [5], some Meromorphic functions that were equivalent to the Riemann zeta function were presented, in which Eq. (5) was given as:

$$\zeta_E(s) = \frac{(s + 2n)}{s - 1}(ks^2 - ks + B(t_n)), \quad (6)$$

or

$$\zeta_E(s) = \frac{(s + 2n)}{e^(s - 1)}(ks^2 - ks + B(t_n)), \quad (7)$$

for $k = 4$, provided that $B(t_n)$ was also known. The authors in Ref. [5] transformed Eqs. (6) and (7) into matrices whose Eigenvalues were the trivial and nontrivial spectral points of the Riemann zeta function provided that;

$$B(n) = 800.162 + 968.548n^{v(n)}, \quad (8)$$

or

$$B(t_n) = 1 + kt_n^2, \quad (9)$$

such that $B(n) = B(t_n)$.

2.1. THE LINK BETWEEN THE ACF FROM EQE

By using the method of discretization on the structured Eq. (5), it becomes:

$$\gamma(s) = \sum_{n\geq 1}^{\infty}(\zeta_E(Z)) = \sum_{n\geq 1}^{\infty}\left[2n\left(1+\frac{s}{2n}\right)(ks^2 - ks + B(t_n))\right]. \quad (10)$$

By applying the method of partial summation [15] on Eq. (10), the resulting equation is given to be:

$$\gamma(s) = \sum_{d\leq n}\left[2n(ks^2 - ks + B(t_n))\right]\sum_{q\leq n/d}\left(1+\frac{s}{2n}\right), \quad (11)$$

where $d = 2n(ks^2 - ks + B(t_n))$ and $q = \left(1 + \frac{s}{2n}\right)$. By using Eq. (10), Eq. (11) can be written as :

$$= -\frac{s}{2}\Gamma(s/2)(s-1)\left[\sum_{d\leq n}2n\left(ks+\frac{B(t_n)}{(s-1)}\right)\right]. \quad (12)$$

By introducing $\pi^{-s/2}\pi^{s/2} = 1$, into Eq. (12), it becomes:

$$\gamma(s) = \phi(s)\left(\sum_{d\leq n}2n\left[\left(ks+\frac{B(t_n)}{(s-1)}\right)\pi^{s/2}\right]\right), \quad (13)$$

where $\phi(s) = -s/2(s - 1)\pi^{-s/2}\Gamma(s/2)$. Using the principle of partial summation on the series in Eq. (12) so that the summation is distributed over the components of the series, Eq. (12) becomes:

$$\beta(s) = \phi(s)\left(\pi^{s/2}\sum_{r\leq n}\left(ks+\frac{B(t_n)}{(s-1)}\right)\sum_{b\leq\frac{n}{r}}2n\right), \quad (14)$$

where $r = \left(ks + \frac{B(t_n)}{(s-1)}\right)\pi^{s/2}$, $b = 2n$. Eq. (14) shall be used shall be later in this paper.

2.2. DERIVATION OF ACF FROM EQ. (14)

The expression [11]

$$\left(\frac{1}{p^s-1}\right)\sum_{n=0}^{\infty}\frac{1}{p^{ns}}=-1, \tag{15}$$

allows us to write $\phi(s)$ as:

$$\phi(s)=\left(\frac{1}{p^s-1}\right)\sum_{n=0}^{\infty}\frac{1}{p^{ns}}\left[\frac{s}{2}(s-1)\pi^{-s/2}\gamma(s/2))\right]. \tag{16}$$

By substituting Eq. (16) into (12), $\gamma(s)$ can be written as:

$$\gamma(s)\left[\left(\frac{1}{p^s-1}\right)\pi^{s/2}\sum_{d\le n}2n[F(t,s)]\right]^{-1} = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\sum_{n=0}^{\infty}\frac{1}{p^{ns}}, \tag{17}$$

where

$$F(t,s)=\left[\left(ks+\frac{B(t_n)}{(s-1)}\right)\pi^{s/2}\right]. \tag{18}$$

Multiplication of Eq. (17) over prime numbers will give;

$$\gamma(s)\prod_p\left[\left(\frac{1}{p^s-1}\right)\pi^{s/2}\sum_{d\le n}2n[F(t,s)]\right]^{-1} = \frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\prod_p\sum_{n\ge 1}\frac{1}{p^{ns}}. \tag{19}$$

Conclusively, the RHS of Eq. (19) is the same as the ACFR, while the LHS is an equivalent of the RHS.

3. NUMERICAL ESTIMATES OF $B(T_N)$

The tables in Appendix show the values of $B(t_n)$ and the corresponding values of n, $v(n)$ for $k = 3, 4, 5\cdots 10$, for which the LHS of Eq. (19) equals its RHS. We proceed to derive an expression for $B(t_n)$by using Eqs. (8) and (9). Let $B(t_n) = B(n)$, then,

$$800.162+968.548n^{v(n)}=1+kt_n^2. \tag{20}$$

This allows us to obtain the zeros of the ACF of the Riemann zeta function as:

$$t_n=\pm\sqrt{\frac{800.162+968.548n^{v(n)}}{k}}. \tag{21}$$

Provided that n and $v(n)$ are as obtained in the tables in the appendix, and $k = 4$. Chudnovsky & Seymour [5] obtained the following for the generation of the zeros of the ACFR:

$$B(t_n)=\frac{ks(s-1)\sigma}{\tau-\vartheta}, \tag{22}$$

where

$$\sigma=\left(\frac{s}{2}\pi^{-s/2}\Gamma(s/2)\left(\frac{1}{p^s-1}\right)\sum_{n=0}^{\infty}\frac{1}{p^{ns}}-1\right). \tag{23}$$

such that;

$$\upsilon=\left(\frac{s}{2}\pi^{-s/2}\Gamma(s/2)\left(\frac{1}{p^s-1}\right)\sum_{n=0}^{\infty}\frac{1}{p^{ns}}\right), \tag{24}$$

and

$$\tau=\left[\frac{1}{2(s-1)}+sN(s)\right]\prod_p\left(\frac{1}{p^s-1}\right), \tag{25}$$

where $N(s)=\sum_{n=1}^{\infty}\left[e^{-n^2\pi}\left(\frac{1^{(\frac{s}{2}-1)}}{2n^2\pi+s-2}+\frac{1^{(\frac{s+1}{2})}}{2n^2\pi-s-2}\right)\right]$ [12]. He pointed out that the above definition is the same as obtained in Eqs. (4)-(7) [6, 7]. He was able to obtain a general equation for the zeros of the analytic continuation formula from Eq. (6) as;

$$B(t_n)=1+kt_n^2; k=1,2,3,\cdots. \tag{26}$$

By which Eq. (19) holds as :

$$t_n=\left(\frac{B(t_n)-1}{k}\right)^{1/2}. \tag{27}$$

Again from:

$$1+kt_n^2=\frac{ks(s-1)\sigma}{\tau-\vartheta}; k=1,2,3,\cdots, \tag{28}$$

such that

$$t_n=\pm\left(\frac{s(s-1)\sigma}{\tau-\vartheta}-\frac{1}{k}\right)^{\frac{1}{2}}; k=1,2,3,\cdots. \tag{29}$$

The k value can hold for any integer, depending on the pattern of choice.

4. PROOF OF THE EQUIVALENCE OF THE LHS AND THE RHS OF EQ. (20)

It has been shown that the analytic continuation formula of the Riemann Zeta function can be obtained from Euler's quadratic equation, and that the Riemann Zeta written as (12) and (19), provided $B(t_n)$ holds as defined a Function can be above. For the LHS of Eq. (19) to be equal to (30), one set;

$$\epsilon_e=\gamma(s)\prod_p\left[\left(\frac{1}{p^s-1}\right)\sum_{d\le n}2nF(t,s)\right]^{-1} \tag{30}$$

$$\epsilon_e=\gamma(s)R(t,s)\prod_p\left[\left(\frac{1}{p^s-1}\right)\right]^{-1}, \tag{31}$$

where $R(t,s)=\prod_p\left[2n\sum_{d\le n}F(t,s)\right]^{-1}$. Then we can also write (31) as the RHS of (19) such that

$$\gamma(s)R(t,s)\zeta(s)=\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{32}$$

where $\epsilon(s)=\epsilon_e$. From the Nachlass of Riemann [10], the ACF, $\epsilon(s)$, is also defined as

$$\epsilon(s)=\frac{1}{2}+\frac{s}{2}(s-1)J(x,s), \tag{33}$$

where

$$J(x,s)=\int_1^{\infty}\psi(x)\left(x^{\frac{s}{2}-1}+x\frac{(s+1)}{2}\right)dx, \tag{34}$$

and

$$\psi(x)=\sum_{n=1}^{\infty}e^{-n^2nx}. \tag{35}$$

Since it has been shown that we can set Eq. (33) to be equal to the RHS of Eq. (32);

$$\frac{1}{2}+\frac{s}{2}(s-1)J(x,s)=\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\prod_{p}\sum_{n\geq 1}^{\infty}\frac{1}{p^{ns}}, \quad (36)$$

or as

$$\frac{1}{2}+\frac{s}{2}(s-1)J(x,s)=\gamma(s)\prod_{p}\left[\left(\frac{1}{p^{s}}-1\right)R(t,s)\right]^{-1}, \quad (37)$$

where

$$R(t,s)=\pi^{s/2}\sum_{d\leq n}\left[2n\left(ks+\frac{B(t_n)}{s-1}\right)\right]. \quad (38)$$

The evaluation of the Intergrades in Eqs (34) and (36) will give

$$J(x,s)=2N(s), \quad (39)$$

where

$$N(s)=\sum_{n=1}^{\infty}\left[e^{-n^2\pi}\left(\frac{1^{(\frac{s}{2}-1)}}{2n^2\pi+s-2}+\frac{1^{(\frac{s+1}{2})}}{2n^2\pi-s-2}\right)\right]. \quad (40)$$

By using Eq. (39) in (37), we write Eq. (37) as

$$\frac{1}{2}+\frac{s}{2}(s-1)N(s)=\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\prod_{p}\sum_{n\geq 1}^{\infty}\frac{1}{p^{ns}}, \quad (41)$$

and

$$\frac{1}{2}+s(s-1)N(s)=\gamma(s)\prod_{p}\left[\left(\frac{1}{p^{s}}-1\right)R(t,s)\right]^{-1}. \quad (42)$$

By Eqs. (41) and (42), we obtain new definitions of the $\zeta(s)$ as:

$$\left[\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\right]^{-1}\left[\frac{1}{2}+s(s-1)N(s)\right] =\prod_{p}\sum_{n\geq 1}^{\infty}\frac{1}{p^{ns}}, \quad (43)$$

and

$$\gamma(s)^{-1}\prod_{p}R(t,s)\left[\frac{1}{2}+s(s-1)N(s)\right]=\prod_{p}\left[\left(\frac{1}{p^{s}}-1\right)\right]^{-1}. \quad (44)$$

Equating Eqs. (43) and (44), we obtain:

$$\left[\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\right]^{-1}\left[\frac{1}{2}+s(s-1)N(s)\right] =\gamma(s)^{-1}\prod_{p}R(t,s)\left[\frac{1}{2}+s(s-1)N(s)\right], \quad (45)$$

such that:

$$\gamma(s)=\left[\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(s/2)\right]^{-1}\prod_{p}R(t,s), \quad (46)$$

and

$$R(t,s)=\pi^{s/2}\sum_{d\leq n}\left[2n\left(ks+\frac{B(t_n)}{s-1}\right)\right]. \quad (47)$$

5. CONCLUSION

At this point, insight into the numerical values of $F(t,s)$ and $R(t,s)$ can be obtained by substituting the expression for $B(t_n)$ and s as defined in Eqs. (8), (9) and (22), for any desired value of k. Conclusively, new representations for the ACF and the appropriate numerical tables for the derived parameters have been presented in this paper. One of the applications of the zeros of the Riemann zeta function is in locating the positions of prime numbers. For instance, the first zero is 14.134725142; the integer part implies that there are prime numbers lesser than 14: 2, 3, 5,7,11 and 13. The second zero of the ACF is 21.022039639; the integer part is 21, which indicates that between 14.134725142 and 21.022039639, there are 17 and 19 as prime numbers. Between 21.022039639 and 25.010857580, we have 23 as a prime number. Between 25.010857580 and 30.424876126, we have 29 as a prime number. 31 is the prime number that lies between 30.424876126 and 32.935061588. 37 is the only prime between 32.935061588 and 37.586178159. So, sometimes the integer part of the zero will be the desired prime number. To mention a few, between 37.586178159 and 40.918719012, 39 is the prime number. The other implications of the integer and the decimal parts shall be explained in subsequent publications.

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APPENDIX

Tables 1-8 show the numerical values of $B(t_n)$ with the corresponding Zeros of the RACF (the non trivial zeros of the Riemann Zeta function).

Table 1. $B(t_n) = 600.1215 + 726.5416j^{v(n)}$; $k = 3$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	600.1215	14.134725142
1	1.00010000000	1326.66675	21.022039639
2	0.81419300000	1877.37975	25.010857580
3	0.99933800001	2827.76975	30.424876126
4	0.93486800000	3254.90475	32.935061588
5	1.00115130000	4238.91225	37.586178159
6	1.00828555000	5023.774725	40.918719012
7	0.99466219910	5632.455837	43.327073281
8	1.03990730100	6914.233533	48.005150881
9	1.02010197400	7433.053196	49.773832478
10	1.03192399600	8418.314873	52.970321478
11	1.04771729700	9559.286633	56.446247697
12	1.05392419200	10566.9649	59.347044003
13	1.04143138700	11102.26583	60.831778525
14	1.066646479101	12719.68018	65.112544048

Table 2. $B(t_n) = 800.162 + 968.548j^{v(n)}$; $k = 4$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	800.162	14.134725142
1	1.00010000000	1768.71	21.022039639
2	0.81419300000	2503.173	25.010857580
3	0.99933800001	3703.693	30.424876126
4	0.93486800000	4339.873	32.935061588
5	1.00115130000	5651.883	37.586178159
6	1.00828555000	6698.3663	40.918719012
7	0.99466219910	7509.941116	43.327073281
8	1.03990730100	9218.978044	48.005150881
9	1.02010197400	9910.737595	49.773832478
10	1.03192399600	11224.41983	52.970321478
11	1.04771729700	12745.71551	56.446247697
12	1.05392419200	14089.28653	59.347044003
13	1.04143138700	14803.0210011	60.831778525
14	1.066646479101	16959.57357	65.112544048

Table 3. $B(t_n) = 1000.2025 + 1210.55075j^{v(n)}$; $k = 5$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	1000.2025	14.134725142
1	1.00010000000	2210.75325	21.022039639
2	0.81419300000	3128.96625	25.010857580
3	0.99933800001	4579.61625	30.424876126
4	0.93486800000	5424.84125	32.935061588
5	1.00115130000	7064.85375	37.586178159
6	1.00828555000	8372.957875	40.918719012
7	0.99466219910	9387.426395	43.327073281
8	1.03990730100	11523.72256	48.005150881
9	1.02010197400	12388.42199	49.773832478
10	1.03192399600	14030.52479	52.970321478
11	1.04771729700	15932.14439	56.446247697
12	1.05392419200	17611.60816	59.347044003
13	1.04143138700	18503.77639	60.831778525
14	1.066646479101	21199.46696	65.112544048

Table 4. $B(t_n) = 1200.2430 + 1452.5535j^{v(n)}$; $k = 6$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	1200.2430	14.134725142
1	1.00010000000	2652.7965	21.022039639
2	0.81419300000	3754.7595	25.010857580
3	0.99933800001	5455.5395	30.424876126
4	0.93486800000	6509.8095	32.935061588
5	1.00115130000	8477.8245	37.586178159
6	1.00828555000	10047.54945	40.918719012
7	0.99466219910	11264.91167	43.327073281
8	1.03990730100	13828.46070	48.005150881
9	1.02010197400	14866.10639	49.773832478
10	1.03192399600	16836.62975	52.970321478
11	1.04771729700	19118.57327	56.446247697
12	1.05392419200	21133.92979	59.347044003
13	1.04143138700	22204.53167	60.831778525
14	1.066646479101	25439.36026	65.112544048

Table 5. $B(t_n) = 1400.2835 + 1694.5563j^{v(n)}$; $k = 7$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	1400.2835	14.134725142
1	1.00010000000	1394.83975	21.022039639
2	0.81419300000	4380.55275	25.010857580
3	0.99933800001	6231.46275	30.424876126
4	0.93486800000	7594.77775	32.935061588
5	1.00115130000	9890.79525	37.586178159
6	1.00828555000	11722.14103	40.918719012
7	0.99466219910	13142.39695	43.327073281
8	1.03990730100	16133.21156	48.005150881
9	1.02010197400	17343.79079	49.773832478
10	1.03192399600	19642.73471	52.970321478
11	1.04771729700	22305.00215	56.446247697
12	1.05392419200	24656.25142	59.347044003
13	1.04143138700	25905.28695	60.831778525
14	1.066646479101	29697.25365	65.112544048

Table 6. $B(t_n) = 1600.324 + 1936.559j^{v(n)}$; $k = 8$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	1600.324	14.134725142
1	1.00010000000	3536.883	21.022039639
2	0.81419300000	5006.346	25.010857580
3	0.99933800001	7207.386	30.424876126
4	0.93486800000	8679.746	32.935061588
5	1.00115130000	11303.766	37.586178159
6	1.00828555000	13396.73261	40.918719012
7	0.99466219910	15019.88223	43.327073281
8	1.03990730100	18437.95607	48.005150881
9	1.02010197400	19821.47519	49.773832478
10	1.03192399600	22448.83967	52.970321478
11	1.04771729700	25491.43103	56.446247697
12	1.05392419200	28178.57305	59.347044003
13	1.04143138700	29606.04223	60.831778525
14	1.066646479101	33919.14004	65.112544048

Table 7. $B(t_n) = 1800.3645 + 1878.56175j^{v(n)}$; $k = 9$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	1800.3645	14.134725142
1	1.00010000000	3678.92625	21.022039639
2	0.81419300000	5632.13925	25.010857580
3	0.99933800001	8083.30925	30.424876126
4	0.93486800000	9764.71425	32.935061588
5	1.00115130000	12716.73675	37.586178159
6	1.00828555000	15071.32419	40.918719012
7	0.99466219910	16897.16751	43.327073281
8	1.03990730100	20742.70058	48.005150881
9	1.02010197400	22299.15959	49.773832478
10	1.03192399600	25254.94463	52.970321478
11	1.04771729700	28677.85991	56.446247697
12	1.05392419200	31700.89468	59.347044003
13	1.04143138700	33306.79751	60.831778525
14	1.066646479101	38159.04043	65.112544048

Table 8. $B(t_n) = 2000.405 + 2120.5645j^{v(n)}$; $k = 10$

J(v)	v(n)	$B(t_n)$	Zeros of RACF
0	0	2000.405	14.134725142
1	1.00010000000	2120.9695	21.022039639
2	0.81419300000	6257.9325	25.010857580
3	0.99933800001	8959.2325	30.424876126
4	0.93486800000	10849.6825	32.935061588
5	1.00115130000	14129.7075	37.586178159
6	1.00828555000	16745.91577	40.918719012
7	0.99466219910	18774.65279	43.327073281
8	1.03990730100	23047.44509	48.005150881
9	1.02010197400	24776.84399	49.773832478
10	1.03192399600	28061.04959	52.970321478
11	1.04771729700	31864.28899	56.446247697
12	1.05392419200	35223.21631	59.347044003
13	1.04143138700	37007.55279	60.831778525
14	1.066646479101	42398.93382	65.112544048